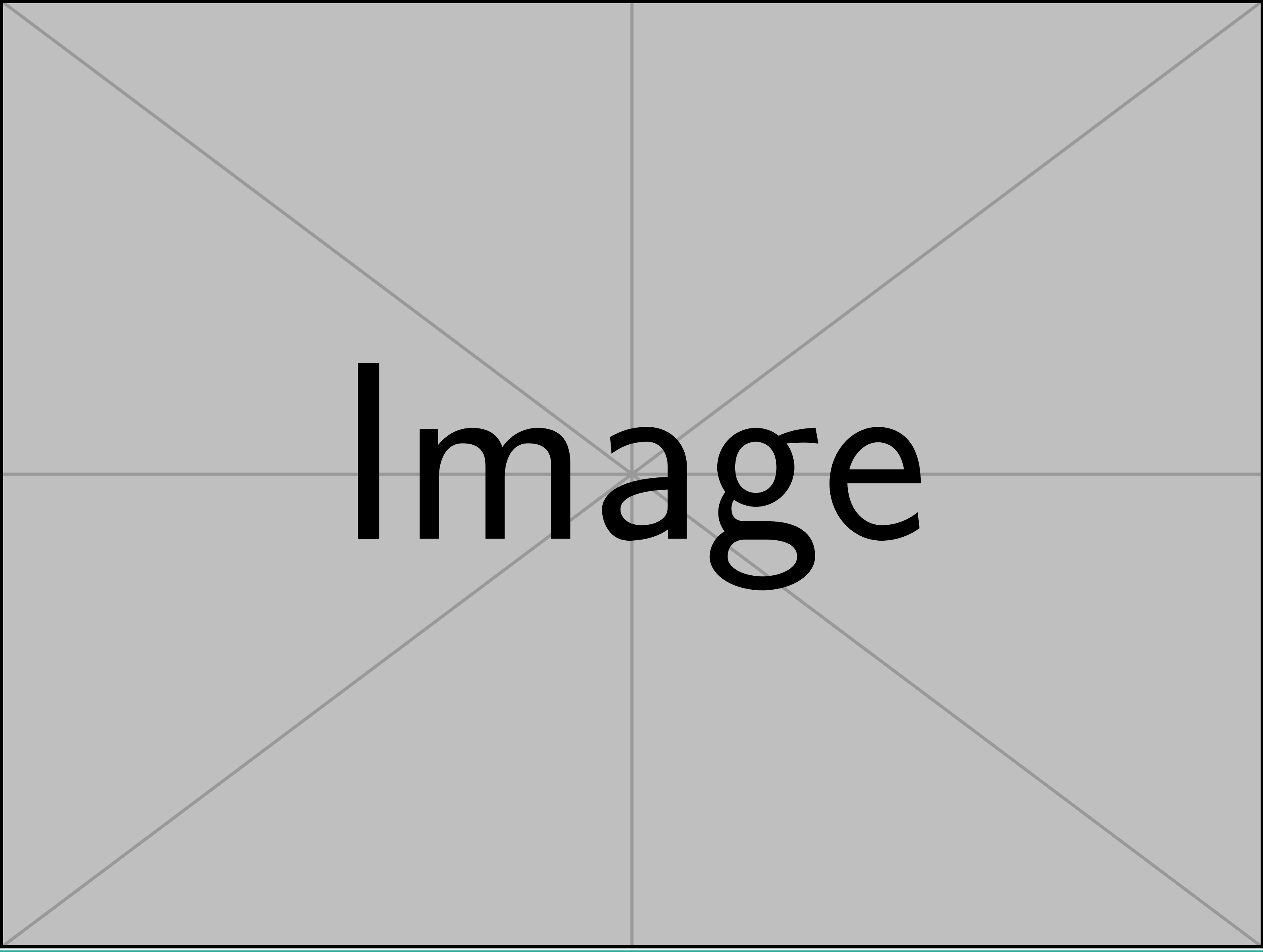

Notes on Mathematics II

Complex & Vector Analysis

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Victory won't come to us unless we go to it.

Contents

Chapter 1 Fourier Series

Introduction

- *Periodic Function*
- *Fourier Series*
- *Even & Odd Functions*

1.1 Periodic function

Definition 1.1

A function $f(x)$ is said to be periodic *if $f(x+T)=f(x)$ for all x, where the period is T.* ♣

Example 1.1 $\sin(x+T)=\sin(x)$.

1.2 Fourier Series

Definition 1.2

If a function $f(x)$ is defined on an interval $(-\pi,\pi)$ and $f(x)$ is periodic and $f(x)$ is piecewise continuous in $(-\pi,\pi)$ then under these three conditions the Fourier series of

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right),$$

where,

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx$$

♣

1.3 Even & Odd Functions

Definition 1.3

A function $f(x)$ is said to be even *if $f(-x)=f(x)$.* ♣

Example 1.2

$$\cos(-x)=\cos x.$$

Definition 1.4

A function $f(x)$ is said to be odd *if $f(-x)=-f(x)$.* ♣

Example 1.3

$$\sin(-x)=-\sin x.$$

Note *For product*

1. *even* $\times$ *even* = *even*
2. *even* $\times$ *odd* = *odd*
3. *odd* $\times$ *odd* = *even*

Proposition 1.1

$$\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(x) \text{is even} \\ 0, & \text{if } f(x) \text{is odd.} \end{cases}$$

Chapter 1 Exercise

1. Define even function and odd function.

Chapter 2 Fourier Integral

Introduction

- Fourier Integral
- Different forms of Fourier Integral
- Fourier Sine and Cosine Integral

2.1 Fourier Integral

Definition 2.1

If $f(x)$ is piecewise continuous on every interval $(-l, l)$ and at a point of discontinuity x_0, $f(x)$ is given by $\frac{f(x_{0-}+f(x_{0+})}{2}$, and also if $\int_{-\infty}^{\infty} |f(t)|dt$ is finite, then for every $\lambda > 0$, the Fourier integral is given by

$$f(x) = \int_0^\infty (A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x)\, d\lambda \tag{2.1}$$

where

$$A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\lambda t dt \tag{2.2}$$

$$B(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\lambda t dt \tag{2.3}$$

♣

Problem 2.1 Find the Fourier integral of

$$f(x) = \begin{cases} -2, & 0 \le x \le 0 \\ 1, & -1 < x < 0 \\ 0, & |x| > 1. \end{cases}$$

Solution *Here, $f(x)$ is piecewise continuous on $(-\infty, \infty)$, and*

$$\begin{aligned}\int_{-\infty}^{\infty} |f(t)|dt &= \int_{-\infty}^{-1} |f(t)|dt + \int_{-1}^{0} |f(t)|dt + \int_{0}^{1} |f(t)|dt + \int_{1}^{\infty} |f(t)|dt \\ &= 0 + \int_{-\infty}^{\infty} 2dt + \int_{-\infty}^{\infty} 1dt + 0 = [2t]_{-1}^{0} + [t]_0^1 = 2 + 1 = 3\end{aligned}$$

which is finite. So, it is possible to find Fourier integral of $f(x)$. Now

$$\begin{aligned}A(\lambda) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\lambda t dt \\ &= \frac{1}{\pi}\left(\int_{-1}^{0} f(t)\cos\lambda t dt + \int_0^1 f(t)\cos\lambda t dt\right) \\ &= \frac{1}{\pi}\left(\int_{-1}^{0} (-2t)\cos\lambda t dt + \int_0^1 1\cos\lambda t dt\right) \\ &= \frac{1}{\pi}\left(\left[\frac{-2\sin\lambda t}{\lambda}\right]_{-1}^{0} + \left[\frac{\sin\lambda t}{\lambda}\right]_0^1\right) \\ &= \frac{1}{\pi}\left(\frac{-2\sin\lambda}{\lambda} + \frac{\sin\lambda}{\lambda}\right) = \frac{1}{\pi}\left(\frac{-\sin\lambda}{\lambda}\right) = -\frac{\sin\lambda}{\pi\lambda}\end{aligned}$$

Again,

$$\begin{aligned}
B(\lambda) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\lambda t dt \\
&= \frac{1}{\pi}\left(\int_{-1}^{0} f(t)\sin\lambda t dt + \int_{0}^{1} f(t)\sin\lambda t dt\right) \\
&= \frac{1}{\pi}\left(\int_{-1}^{0} (-2t)\sin\lambda t dt + \int_{0}^{1} 1\sin\lambda t dt\right) \\
&= \frac{1}{\pi}\left(\left[\frac{-2(-\cos\lambda t)}{\lambda}\right]_{-1}^{0} + \left[\frac{-\cos\lambda t}{\lambda}\right]_{0}^{1}\right) \\
&= \frac{1}{\pi}\left(\frac{2}{\lambda} - \frac{2\cos\lambda}{\lambda} - \frac{\cos\lambda}{\lambda} + \frac{1}{\lambda}\right) = \frac{1}{\pi}\left(\frac{3}{\lambda} - \frac{3\cos\lambda}{\lambda}\right) = \frac{3}{\pi\lambda}(1-\cos\lambda)
\end{aligned}$$

The Fourier integral of $f(x)$ is

$$\begin{aligned}
f(x) &= \int_{0}^{\infty}\left(-\frac{\sin\lambda}{\pi\lambda}\cos\lambda x + \frac{3}{\pi\lambda}(1-\cos\lambda)\sin\lambda x\right) d\lambda \\
&= \frac{1}{\pi}\int_{0}^{\infty}\left(-\frac{\sin\lambda}{\lambda}\cos\lambda x + \frac{3}{\lambda}(1-\cos\lambda)\sin\lambda x\right) d\lambda
\end{aligned}$$

2.2 Fourier Cosine and Sine Integral

If $f(x)$ is even function then from (2.2)-(2.3), we have

$$\begin{aligned}
A(\lambda) &= \frac{2}{\pi}\int_{0}^{\infty} f(t)\cos\lambda t dt \\
B(\lambda) &= 0
\end{aligned}$$

Putting these values in (2.1), we get Fourier integral for even function, or Fourier Cosine integral

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty}\left(\int_{0}^{\infty} f(t)\cos\lambda t dt\right)\cos\lambda x d\lambda$$

Similarly, if $f(x)$ is odd function then from (2.2)-(2.3), we have

$$\begin{aligned}
A(\lambda) &= 0 \\
B(\lambda) &= \frac{2}{\pi}\int_{0}^{\infty} f(t)\sin\lambda t dt
\end{aligned}$$

Putting these values in (2.1), we get Fourier integral for odd function, or Fourier Sine integral

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty}\left(\int_{0}^{\infty} f(t)\sin\lambda t dt\right)\sin\lambda x d\lambda$$

Problem 2.2 Find the Fourier integral of e^{-x}; $x > 0$.

Or, Show that

$$\int_{0}^{\alpha}\frac{\cos\lambda x}{\lambda^2+1}d\lambda = \frac{\pi}{2}e^{-x}; \qquad x \geq 0$$

Solution *Here, $f(x)$ is continuous on $(0,\infty)$, and*

$$\int_{0}^{\infty}|f(t)|dt = \int_{0}^{\infty} e^{-t}dt = [-e^{-t}]_{0}^{\infty} = [0+1] = 1$$

which is finite. So, it is possible to find Fourier integral of $f(x)$. Now

$$\begin{aligned} A(\lambda) &= \frac{2}{\pi}\int_0^\infty f(t)\cos\lambda t dt \\ &= \frac{2}{\pi}\int_0^\infty e^{-t}\cos\lambda t dt \\ &= \frac{2}{\pi}\left[\frac{e^{-t}(-\cos\lambda t+\lambda\sin\lambda t)}{1+\lambda^2}\right]_0^\infty \\ &= \frac{2}{\pi}\left[0-\frac{-1}{1+\lambda^2}\right] \\ &= \frac{2}{\pi(1+\lambda^2)}, \end{aligned}$$

And

$$B(\lambda) = 0$$

The Fourier integral of e^{-x} is

$$\begin{aligned} e^{-x} &= \int_0^\infty \frac{2}{\pi(1+\lambda^2)}\cos\lambda x d\lambda \\ &= \frac{2}{\pi}\int_0^\infty \frac{\cos\lambda x}{1+\lambda^2} d\lambda \\ \implies \int_0^\infty \frac{\cos\lambda x}{1+\lambda^2} d\lambda &= \frac{\pi e^{-x}}{2} \end{aligned}$$

2.3 Different forms of Fourier Integrals

Proposition 2.1

Fourier integral formula (2.1) *also can be rewritten as follows*

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty f(t)\cos\lambda(t-x) dt d\lambda. \tag{2.4}$$

Chapter 2 Exercise

1. Define Fourier integral.
2. Write down the Fourier integral formula for even function.
3. Write down Fourier Cosine and Sine integrals.
4. Find the Fourier integral of

$$f(x) = \begin{cases} -2, & 0 \le x \le 0 \\ 1, & -1 < x < 0 \\ 0, & |x| > 1. \end{cases}$$

5. Find the Fourier integral of e^{-x}; $x > 0$.
6. Show that

$$\int_0^\alpha \frac{\cos\lambda x}{\lambda^2+1} d\lambda = \frac{\pi}{2}e^{-x}; \qquad x \ge 0$$

Chapter 3 Fourier Transform

Introduction

- *Fourier Transform*
- *Property of Fourier Transform*
- *Convolution*
- *Convolution theorem of Fourier Transform*
- *Fourier Sine Transform*
- *Fourier Cosine Transform*

3.1 Fourier Transform

Definition 3.1

If a function $f(x)$ is defined on $(-\infty, \infty)$ it is continuous and piece-wise smooth, $f(t) \to 0$ when $|t| \to \alpha$ and $f(x)$ is absolutely integrable then the Fourier transform *of $f(x)$ denoted by $F(\alpha)$ is defined by*

$$\mathcal{F}[f(x)] = F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\alpha t} dt. \tag{3.1}$$

The inverse of $F(\alpha)$ denoted by $\mathcal{F}^{-1}[F(\alpha)]$ is given by

$$\mathcal{F}^{-1}[F(\alpha)] = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha) e^{-i\alpha x} d\alpha \tag{3.2}$$

♣

3.2 Properties of Fourier Transform

Some useful properties of Fourier transforms are as follows:

Fourier transform is linear	$\mathcal{F}[af(t) + bg(t)] = a\mathcal{F}[f(t)] + b\mathcal{F}[g(t)] = aF(\alpha) + bG(\alpha)$
Shifting property	$\mathcal{F}[f(t-c)] = e^{i\alpha c}\mathcal{F}[f(t)] = e^{i\alpha c}F(\alpha)$
Scaling property	$\mathcal{F}[f(ct)] = \frac{1}{c}\mathcal{F}\left[f(\frac{t}{c})\right] = \frac{1}{c}F(\frac{t}{c})$
Differentiation	$\mathcal{F}[f'(t)] = -i\alpha\mathcal{F}[f(t)] = -i\alpha F(\alpha)$
Modulation property	$f(x)\cos ax = \frac{1}{2}[F(\alpha + a) + F(\alpha - a)]$

Proposition 3.1

If $f(x)$ has the Fourier transform $F(\alpha)$, then $f(x)\cos ax$ has the Fourier transform

$$\frac{1}{2}\left(F(\alpha + a) + F(\alpha - a)\right).$$

♠

Proof

$$\begin{aligned}
\mathcal{F}[f(x)\cos ax] &= \int_{-\infty}^{\infty} f(x)\cos ax e^{-i\alpha x} dx \\
&= \int_{-\infty}^{\infty} f(x)\frac{1}{2}\left(e^{iax} + e^{-iax}\right) e^{-i\alpha x} dx \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \left[e^{-i(\alpha-a)x} f(x) + e^{-i(\alpha+a)x} f(x)\right] dx \\
&= \frac{1}{2}\left[\int_{-\infty}^{\infty} e^{-i(\alpha-a)x} f(x) dx + \int_{-\infty}^{\infty} e^{-i(\alpha+a)x} f(x) dx\right]
\end{aligned}$$

Hence,

$$\mathcal{F}\left[f(x)\cos ax\right] = \frac{1}{2}\left(F(\alpha + a) + F(\alpha - a)\right)$$

3.3 Convolution

Definition 3.2

If two functions $f(x)$ and $g(x)$ are defined on $(-\infty, \infty)$ then the convolution of $f(x)$ and $g(x)$ is denoted by $f * g$, or $f(x) * g(x)$ and is defined by

$$f * g = \int_{-\infty}^{\infty} f(u)g(x-u)du. \tag{3.3}$$

♣

3.4 Convolution of Fourier Transform

Theorem 3.1

The Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of the Fourier transform of $f(x)$ and the Fourier transform of $g(x)$ i.e.

$$\mathcal{F}[f * g] = \mathcal{F}[f] * \mathcal{F}[g]. \tag{3.4}$$

♡

Proof

$$\begin{aligned}
\mathcal{F}[f * g] &= \int_{-\infty}^{\infty} [f * g]e^{i\alpha x}dx \\
&= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(u)g(x-u)du\right]e^{i\alpha x}dx \\
&= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} g(x-u)e^{i\alpha x}dx\right]f(u)du
\end{aligned} \tag{3.5}$$

Let $x - u = v$, then $dx = dv$, $x = u + v$, putting all these values and changing corresponding limits in (3.5), we have

$$\begin{aligned}
\mathcal{F}[f * g] &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} g(v)e^{i\alpha(u+v)}dv\right]f(u)du \\
&= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} g(v)e^{i\alpha v}dv\right]e^{i\alpha u}f(u)du \\
&= \int_{-\infty}^{\infty}\left(\mathcal{F}[g]\right)e^{i\alpha u}f(u)du \\
&= \mathcal{F}[g]\int_{-\infty}^{\infty} e^{i\alpha u}f(u)du \\
\mathcal{F}[f * g] &= \mathcal{F}[f]\mathcal{F}[g]
\end{aligned} \tag{3.6}$$

Problem 3.1 Find the Fourier transform of $f(x) = e^{-|x|}$.

We also have,

$$\begin{aligned}
&= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{x}e^{i\alpha x}dx + \int_{0}^{\infty} e^{-x}e^{i\alpha x}dx\right]\\
&= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{(1+i\alpha)x}dx + \int_{0}^{\infty} e^{-(+1-i\alpha)x}dx\right]\\
&= \frac{1}{\sqrt{2\pi}}\left(\left[\frac{e^{(1+i\alpha)x}}{1+i\alpha}\right]_{-\infty}^{0} + \left[\frac{e^{-(1-i\alpha)x}}{-(1-i\alpha)}\right]_{0}^{\infty}\right)\\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{1+i\alpha}\left(e^{0}-e^{-\infty}\right) + \frac{1}{-(1-i\alpha)}\left(e^{-\infty}-e^{0}\right)\right)\\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{1+i\alpha}(1-0) + \frac{1}{-(1-i\alpha)}(0-1)\right)\\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{1+i\alpha} + \frac{1}{(1-i\alpha)}\right)\\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{1+i\alpha+1-i\alpha}{1-i^2\alpha^2}\right) = \frac{1}{\sqrt{2\pi}}\frac{2}{1+\alpha^2} = \sqrt{\frac{2}{\pi}}\frac{1}{1+\alpha^2} \qquad (3.7)
\end{aligned}$$

Problem 3.2 Let

$$f(x) = \begin{cases} 1-x^2, & |x| \le 1 \\ 0, & |x| > 1, \end{cases} \qquad (3.8)$$

and Hence evaluate

$$\int_{0}^{\infty}\left(\frac{x\cos x - \sin x}{x^3}\right)\cos\frac{x}{2}dx$$

Solution *By the definition of Fourier transform*

$$\begin{aligned}
\mathcal{F}\left[f(x)\right] &= \int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx\\
&= \int_{-\infty}^{-1} f(x)e^{i\alpha x}dx + \int_{-1}^{1} f(x)e^{i\alpha x}dx + \int_{1}^{\infty} f(x)e^{i\alpha x}dx\\
&= 0 + \int_{-1}^{1}(1-x^2)e^{i\alpha x}dx + 0\\
&= \left[(1-x^2)\frac{e^{i\alpha x}}{i\alpha}\right]_{-1}^{1} + 2\int_{-1}^{1} x\frac{e^{i\alpha x}}{i\alpha}dx\\
&= 2\left[\frac{xe^{i\alpha x}}{(i\alpha)^2}\right]_{-1}^{1} - 2\int_{-1}^{1}\frac{e^{i\alpha x}}{(i\alpha)^2}dx\\
&= -\frac{2}{\alpha^2}\left(e^{i\alpha}+e^{-i\alpha}\right) + \frac{2}{i\alpha^3}\left[e^{i\alpha x}\right]_{-1}^{1}\\
&= -\frac{4}{\alpha^2}\cos\alpha + \frac{4}{i\alpha^3}\left[e^{i\alpha}-e^{-i\alpha}\right]\\
&= -\frac{4}{\alpha^2}\cos\alpha + \frac{4}{\alpha^3}\cos\alpha\\
\implies F(\alpha) &= 4\left(\frac{\sin\alpha - \alpha\cos\alpha}{\alpha^3}\right) \qquad (3.9)
\end{aligned}$$

Taking the corresponding inversion formula.

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)e^{-i\alpha x}d\alpha \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 4\left(\frac{\sin\alpha-\alpha\cos\alpha}{\alpha^3}\right)e^{-i\alpha x}d\alpha \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 4\left(\frac{\sin\alpha-\alpha\cos\alpha}{\alpha^3}\right)(\cos\alpha x - i\sin\alpha x)\,d\alpha \\ &= \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\alpha-\alpha\cos\alpha}{\alpha^3}\right)\cos\alpha x d\alpha - \frac{2i}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\alpha-\alpha\cos\alpha}{\alpha^3}\right)\sin\alpha x d\alpha \end{aligned}$$

Equating the real part from both sides

$$f(x) = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\alpha-\alpha\cos\alpha}{\alpha^3}\right)\cos\alpha x d\alpha$$

putting $x=\frac{1}{2}$*, we get*

$$\begin{aligned} \frac{4}{\pi}\int_0^{\infty}\left(\frac{\sin\alpha-\alpha\cos\alpha}{\alpha^3}\right)\cos\frac{\alpha}{2}d\alpha &= \left(1-\frac{1}{4}\right)=\frac{3}{4} \\ \int_0^{\infty}\left(\frac{\sin\alpha-\alpha\cos\alpha}{\alpha^3}\right)\cos\frac{\alpha}{2}d\alpha &= \frac{3\pi}{16} \\ \int_0^{\infty}\left(\frac{\alpha\cos\alpha-\sin\alpha}{\alpha^3}\right)\cos\frac{\alpha}{2}d\alpha &= -\frac{3\pi}{16} \end{aligned}$$

changing variable $\alpha = x$*,*

$$\int_0^{\infty}\left(\frac{x\cos x-\sin x}{x^3}\right)\cos\frac{x}{2}dx = -\frac{3\pi}{16}.$$

3.5 Fourier Sine Transform

Definition 3.3

The infinite Fourier sine transform *of* $F(x)$, $0<x<\infty$ *is defined by*

$$f_s(n) = \int_0^{\infty} F(x)\sin nx dx \tag{3.10}$$

where n *is an integer. The function* $F(x)$ *is then called the inverse Fourier sine transform of* $f_s(n)$ *and we can write*

$$f(x) = \frac{2}{\pi}\int_0^{\infty} f_s(n)\sin nx dn. \tag{3.11}$$

♣

3.6 Fourier Cosine Transform

Definition 3.4

The infinite Fourier cosine transform *of* $F(x)$, $0<x<\infty$ *is defined by*

$$f_c(n) = \int_0^{\infty} F(x)\cos nx dx \tag{3.12}$$

where n *is an integer. The function* $F(x)$ *is then called the inverse Fourier cosine transform of* $f_c(n)$ *and we can write*

$$f(x) = \frac{2}{\pi}\int_0^{\infty} f_c(n)\cos nx dn. \tag{3.13}$$

♣

Problem 3.3 Find the Fourier sine and cosine transform of $e^{-x}; x > 0$.

Solution *From the definition of Fourier sine transform*

$$f_s(n) = \int_0^\infty F(x) \sin nx dx = \int_0^\infty e^{-x} \sin nx dx \tag{3.14}$$

Now, let

$$\begin{aligned} I &= \int e^{-x} \sin nx dx = -e^{-x} \sin nx + n \int e^{-x} \cos nx dx \\ &= -e^{-x} \sin nx - ne^{-x} \cos nx - n^2 \int e^{-x} \sin nx dx \\ &= -e^{-x} \sin nx - ne^{-x} \cos nx - n^2 I \\ \implies (n^2+1)I &= -e^{-x} \sin nx - ne^{-x} \cos nx + C \\ \implies I &= \frac{e^{-x}}{(n^2+1)} (-\sin nx - n \cos nx) + C \end{aligned}$$

Using I in (3.14) *we have,*

$$\begin{aligned} f_s(n) &= \frac{1}{(n^2+1)} \left[-e^{-x} (\sin nx + n \cos nx)\right]_0^\infty \\ &= \frac{1}{(n^2+1)} [-0 + (n)] = \frac{n}{(n^2+1)}. \end{aligned}$$

Similarly, from the definition of Fourier cosine transform

$$f_c(n) = \int_0^\infty F(x) \cos nx dx = \int_0^\infty e^{-x} \cos nx dx \tag{3.15}$$

Now, let

$$\begin{aligned} J &= \int e^{-x} \cos nx dx = -e^{-x} \cos nx - n \int e^{-x} \sin nx dx \\ &= -e^{-x} \cos nx + ne^{-x} \sin nx - n^2 \int e^{-x} \cos nx dx \\ &= -e^{-x} \cos nx + ne^{-x} \sin nx - n^2 J \\ \implies (n^2+1)J &= -e^{-x} \cos nx + ne^{-x} \sin nx + C \\ \implies J &= \frac{e^{-x}}{(n^2+1)} (-\cos nx + n \sin nx) + C \end{aligned}$$

Using J in (3.15) *we have,*

$$\begin{aligned} f_c(n) &= \frac{1}{(n^2+1)} \left[e^{-x} (-\cos nx + n \sin nx)\right]_0^\infty \\ &= \frac{1}{(n^2+1)} [0 + 1] = \frac{1}{(n^2+1)}. \end{aligned}$$

Chapter 3 Exercise

1. If $f(x)$ has the Fourier transform $F(\alpha)$, then $f(x) \cos ax$ has the Fourier transform
$$\frac{1}{2} \left(F(\alpha + a) + F(\alpha - a)\right).$$
2. Find the Fourier transform of $f(x) = e^{-|x|}$.
3. Prove that Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of the Fourier transform of $f(x)$ and $g(x)$.
4. Find the Fourier sine and cosine transform of $e^{-x}; x > 0$.

5. Let

$$f(x) = \begin{cases} 1 - x^2, & |x| \leq 1 \\ 0, & |x| > 1, \end{cases}$$

and Hence evaluate

$$\int_0^\infty \left(\frac{x \cos x - \sin x}{x^3} \right) \cos \frac{x}{2} dx$$

Chapter 4 Finite Fourier Transform

Introduction

- *Finite Fourier Sine Transform*
- *Finite Fourier Cosine Transform*

4.1 Finite Fourier Sine Transform

Definition 4.1

The finite Fourier sine transform *of* $F(x)$, $0 < x < l$ *is defined by*

$$f_s(n) = \int_0^l F(x) \sin \frac{n\pi x}{l} dx \tag{4.1}$$

where n is an integer. The function $F(x)$ is then called the inverse finite Fourier sine transform of $f_s(n)$ and we can write

$$f(x) = \frac{2}{l} \sum_{n=1}^{\infty} f_s(n) \sin \frac{n\pi x}{l}. \tag{4.2}$$

♣

4.2 Finite Fourier Cosine Transform

Definition 4.2

The finite Fourier cosine transform *of* $F(x)$, $0 < x < l$ *is defined by*

$$f_c(n) = \int_0^l F(x) \cos \frac{n\pi x}{l} dx \tag{4.3}$$

where n is an integer. The function $F(x)$ is then called the inverse finite Fourier cosine transform of $f_c(n)$ and we can write

$$f(x) = \frac{2}{l} \sum_{n=1}^{\infty} f_c(n) \cos \frac{n\pi x}{l}. \tag{4.4}$$

♣

Theorem 4.1

For finite Fourier transform

$$F(x) = \frac{2}{l} \sum_{n=1}^{\infty} f_s(n) \sin \frac{n\pi x}{l} \tag{4.5}$$

$$F(x) = \frac{1}{l} f_c(0) + \frac{2}{l} \sum_{n=1}^{\infty} f_c(n) \cos \frac{n\pi x}{l} \tag{4.6}$$

♡

Proof If $F(x)$ be a single valued function $(-l, l)$, then

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right) \tag{4.7}$$

where,

$$a_n = \frac{1}{l}\int_{-l}^{l} F(x)\cos\frac{n\pi x}{l}dx \tag{4.8}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} F(x)\sin\frac{n\pi x}{l}dx \tag{4.9}$$

Let $F(x)$ is odd then $F(x)\cos\frac{n\pi x}{l}$ is also odd then we have,

$$a_n = \frac{1}{l}\int_{-l}^{l} F(x)\cos\frac{n\pi x}{l}dx = 0.$$

Also putting $n = 0$ in (4.8), we get,

$$a_0 = \frac{1}{l}\int_{-l}^{l} F(x)dx = 0.$$

Again if $F(x)$ is odd then $F(x)\sin\frac{n\pi x}{l}$ is even, applying this in (4.9)

$$b_n = \frac{1}{l}\int_{-l}^{l} F(x)\sin\frac{n\pi x}{l}dx = \frac{2}{l}\int_{0}^{l} F(x)\sin\frac{n\pi x}{l}dx$$

Now putting a_0, a_n, and b_n in (4.7), we get

$$\begin{aligned} F(x) &= \sum_{n=1}^{\infty}\left(\frac{2}{l}\left(\int_0^l F(x)\sin\frac{n\pi x}{l}dx\right)\sin\frac{n\pi x}{l}\right) \\ &= \frac{2}{l}\sum_{n=1}^{\infty} f_s(n)\sin\frac{n\pi x}{l} \end{aligned}$$

where

$$f_s(n) = \int_0^l F(x)\sin\frac{n\pi x}{l}dx.$$

Again, let $F(x)$ is even then $F(x)\cos\frac{n\pi x}{l}$ is also even then we have,

$$a_n = \frac{1}{l}\int_{-l}^{l} F(x)\cos\frac{n\pi x}{l}dx = \frac{2}{l}\int_{0}^{l} F(x)\cos\frac{n\pi x}{l}dx.$$

Also putting $n = 0$ in (4.8), we get,

$$a_0 = \frac{1}{l}\int_{-l}^{l} F(x)dx = \frac{2}{l}\int_{0}^{l} F(x)dx.$$

Again if $F(x)$ is even then $F(x)\sin\frac{n\pi x}{l}$ is odd, applying this in (4.9)

$$b_n = \frac{1}{l}\int_{-l}^{l} F(x)\sin\frac{n\pi x}{l}dx = 0.$$

Now putting a_0, a_n, and b_n in (4.7), we get

$$\begin{aligned} F(x) &= \frac{1}{l}\int_0^l F(x)dx + \sum_{n=1}^{\infty}\left(\frac{2}{l}\left(\int_0^l F(x)\cos\frac{n\pi x}{l}dx\right)\cos\frac{n\pi x}{l}\right) \\ &= \frac{1}{l}f_c(0) + \frac{2}{l}\sum_{n=1}^{\infty} f_c(n)\sin\frac{n\pi x}{l} \end{aligned}$$

where

$$f_c(0) = \int_0^l F(x)dx$$

and

$$f_c(n) = \int_0^l F(x)\cos\frac{n\pi x}{l}dx.$$

Problem 4.1 Find the finite Fourier sine transform of

$$F(x) = \cos kx; \qquad 0 < x < \pi.$$

Solution *We know*

$$\begin{aligned}
f_s(n) &= \int_0^\pi F(x)\sin nx dx = \int_0^\pi \cos kx \sin nx dx \\
&= \frac{1}{2}\int_0^\pi (\sin(n+k)x + \sin(n-k)x)\, dx \\
&= \frac{1}{2}\left[-\frac{\cos(n+k)x}{n+k} - \frac{\cos(n-k)x}{n-k}\right]_0^\pi \\
&= \frac{1}{2}\left[-\frac{\cos(n+k)\pi}{n+k} + \frac{1}{n+k} - \frac{\cos(n-k)\pi}{n-k} + \frac{1}{n-k}\right] \\
&= \frac{1}{2}\left[-\frac{(n-k)\cos(n+k)\pi + (n+k)\cos(n-k)\pi}{(n+k)(n-k)} + \frac{n-k+n+k}{(n+k)(n-k)}\right] \\
&= \frac{1}{2}\left[-\frac{n\left(\cos(n+k)\pi + \cos(n-k)\pi\right) - k\left(\cos(n+k)\pi - \cos(n-k)\pi\right)}{n^2-k^2} + \frac{2n}{n^2-k^2}\right] \\
&= \frac{1}{2(n^2-k^2)}\left[-n\left(\cos(n+k)\pi + \cos(n-k)\pi\right) + k\left(\cos(n+k)\pi - \cos(n-k)\pi\right) + n\right] \\
&= \frac{1}{2(n^2-k^2)}\left[-n\cos k\pi\cos n\pi - k\sin k\pi \sin n\pi + n\right] \\
&= \frac{n}{2(n^2-k^2)}\left[1 - \cos k\pi \cos n\pi\right] \\
&= \frac{n}{2(n^2-k^2)}\left[1 - (-1)^n \cos k\pi\right].
\end{aligned}$$

Problem 4.2 Find the finite Fourier sine transform of

$$F(x) = \begin{cases} x: & 0 \le x \le \pi/2 \\ \pi - x: & \pi/2 \le x \le \pi \end{cases}$$

Solution *We know*

$$\begin{aligned}
f_s(n) &= \int_0^\pi F(x)\sin nx dx \\
&= \int_0^{\pi/2} F(x)\sin nx dx + \int_{\pi/2}^\pi F(x)\sin nx dx \\
&= \int_0^{\pi/2} x\sin nx dx + \int_{\pi/2}^\pi (\pi - x)\sin nx dx \\
&= \left[\frac{-x\cos nx}{n}\right]_0^{\pi/2} + \frac{1}{n}\int_0^{\pi/2}\cos nx dx + \left[\frac{(\pi-x)\cos nx}{-n}\right]_{\pi/2}^\pi - \frac{1}{n}\int_{\pi/2}^\pi \cos nx dx \\
&= -\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\left[\sin nx\right]_0^{\pi/2} + \frac{\pi}{2n}\cos\frac{n\pi}{2} - \frac{1}{n^2}\left[\sin nx\right]_{\pi/2}^\pi \\
&= \frac{1}{n^2}\left[\sin\frac{n\pi}{2} - 0\right] - \frac{1}{n^2}\left[0 - \sin\frac{n\pi}{2}\right] \\
f_s(n) &= \frac{2}{n^2}\sin\frac{n\pi}{2}.
\end{aligned}$$

Problem 4.3 Find the finite Fourier cosine transform of $F(x) = 2x; \qquad 0 < x < 4.$

Solution *Here, l=4, so we have,*

$$\begin{aligned} f_c(n) &= \int_0^4 F(x)\cos\frac{n\pi x}{4}dx \\ &= 2\int_0^4 x\cos\frac{n\pi x}{4}dx \\ &= \frac{2\cdot 4}{n\pi}\left[x\sin\frac{n\pi x}{4}\right]_0^4 - \frac{8}{n\pi}\int_0^4 \sin\frac{n\pi x}{4}dx \\ &= 0+\frac{32}{n^2\pi^2}\left[\cos\frac{n\pi x}{4}\right]_0^4 = \frac{32}{n\pi}(\cos n\pi - 1)\,. \end{aligned}$$

Problem 4.4 Find the finite sine and cosine transform of

$$f(x) = \left(1-\frac{x}{\pi}\right)$$

Solution *We have,*

$$\begin{aligned} f_s(n) &= \int_0^\pi f(x)\sin nx dx \\ &= \int_0^\pi \left(1-\frac{x}{\pi}\right)\sin nx dx \\ &= \int_0^\pi \sin nx dx - \frac{1}{\pi}\int_0^\pi x\sin nx dx \\ &= -\frac{1}{n}[\cos nx]_0^\pi + \frac{1}{n\pi}[x\cos nx]_0^\pi - \frac{1}{\pi}\int_0^\pi \cos nx dx \\ &= -\frac{1}{n}[\cos n\pi - 1] + \frac{1}{n\pi}[\pi\cos n\pi - 0] - \frac{1}{n^2\pi}[\sin nx]_0^\pi \\ &= \frac{1}{n} - 0 = \frac{1}{n}. \end{aligned}$$

Also we have,

$$\begin{aligned} f_c(n) &= \int_0^\pi f(x)\cos nx dx \\ &= \int_0^\pi \left(1-\frac{x}{\pi}\right)\cos nx dx \\ &= \int_0^\pi \cos nx dx - \frac{1}{\pi}\int_0^\pi x\cos nx dx \\ &= \frac{1}{n}[\sin nx]_0^\pi - \frac{1}{n\pi}[x\sin nx]_0^\pi + \frac{1}{\pi}\int_0^\pi \sin nx dx \\ &= \frac{1}{n}[\sin n\pi - 0] - \frac{1}{n\pi}[\pi\sin n\pi - 0] - \frac{1}{n^2\pi}[\cos nx]_0^\pi \\ &= -\frac{1}{n^2\pi}[\cos n\pi - 1] = -\frac{1}{n^2\pi}[(-1)^n - 1] = \frac{1}{n^2\pi}[1-(-1)^n]\,. \end{aligned}$$

Chapter 4 Exercise

1. Define finite Fourier sine transform.
2. Prove that for finite Fourier transform

   (a).

$$F(x) = \frac{2}{l}\sum_{n=1}^{\infty} f_s(n)\sin\frac{n\pi x}{l}$$

(b).

$$F(x) = \frac{1}{l} f_c(0) + \frac{2}{l} \sum_{n=1}^{\infty} f_c(n) \cos \frac{n\pi x}{l}$$

3. Find the finite Fourier sine transform of

$$F(x) = \begin{cases} x : & 0 \le x \le \pi/2 \\ \pi - x : & \pi/2 \le x \le \pi \end{cases}$$

4. Find the finite sine and cosine transform of

$$f(x) = \left(1 - \frac{x}{\pi}\right)$$

5. Find the finite Fourier sine transform of

$$F(x) = \cos kx; \qquad 0 < x < \pi.$$

6. Find the finite Fourier cosine transform of $F(x) = 2x; \qquad 0 < x < 4.$

Chapter 5 Application of Finite Fourier Transform

Introduction

- *Four formulae related to Boundary Value Problem*
- *Selection of Finite Sine or Cosine Transform*
- *Application of Finite Fourier Transform*

5.1 Four formulae related to Boundary Value Problem

$$
\begin{aligned}
f_c\left\{\frac{\partial U}{\partial x}\right\} &= U(l,t)\cos n\pi - U(0,t) - \frac{n\pi}{l} f_s(U) \\
f_s\left\{\frac{\partial U}{\partial x}\right\} &= -\frac{n\pi}{l} f_c(U) \\
f_s\left\{\frac{\partial^2 U}{\partial x^2}\right\} &= -\frac{n\pi}{l} U(l,t)\cos n\pi + \frac{n\pi}{l} U(0,t) - \frac{n^2\pi^2}{l} f_s(U) \\
f_c\left\{\frac{\partial^2 U}{\partial x^2}\right\} &= U_x(l,t)\cos n\pi - U_x(0,t) + \frac{n^2\pi^2}{l^2} f_c(U)
\end{aligned}
$$

5.2 Selection of Finite Sine or Cosine Transform

We have to choose finite sine or cosine transform by the form of boundary conditions, such that

1. If Dirichlet boundary condition that is boundary conditions are provided for $U(0,t)$ and $U(l,t)$ then choose finite sine transform.
2. For Neumann boundary condition that is boundary conditions are provided for $U_x(0,t)$ and $U_x(l,t)$then choose finite cosine transform.

5.3 Application of Finite Fourier Transform

Problem 5.1 By Fourier transform solve

$$
\begin{aligned}
\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}, \qquad & 0 < x < \pi, t > 0 \\
U(0,t) = U(\pi,t) = 0, \qquad & t > 0, \\
U(x,0) = 2x, \qquad & 0 < x < \pi.
\end{aligned}
$$

Solution *Given,*

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \tag{5.1}$$

Taking sine transform both side of (5.3)

$$\int_0^\pi \frac{\partial U}{\partial t}\sin nx dx = \int_0^\pi \frac{\partial^2 U}{\partial x^2}\sin nx dx \tag{5.2}$$

Let

$$V = V(n,t) = \int_0^\pi U(x,t)\sin nx dx \tag{5.3}$$

Differentiating (5.3) *wrt t, we get*

$$\begin{aligned}
\frac{\partial V}{\partial t}=\int_0^{\pi}\frac{\partial U}{\partial t}\sin nx dx &= \int_0^{\pi}\frac{\partial^2 U}{\partial x^2}\sin nx dx \qquad [Using (5.2)]\\
&= \left[\frac{\partial U}{\partial x}\sin nx\right]_0^{\pi} - n\int_0^{\pi}\frac{\partial U}{\partial x}\cos nx dx\\
&= 0 - n\left[U(x,t)\cos nx\right]_0^{\pi} - n^2\int_0^{\pi}U(x,t)\sin nx dx\\
&= -n^2 V \qquad [Using (5.3)] \qquad (5.4)
\end{aligned}$$

$$\begin{aligned}
\implies \frac{dV}{dt} &= -n^2 V\\
\implies \frac{dV}{V} &= -n^2 dt\\
\implies \ln V &= -n^2 t + \ln C \qquad [Integrating]\\
\implies V &= Ce^{-n^2 t} \qquad [Integrating] \qquad (5.5)
\end{aligned}$$

When $t = 0$ *then from* (5.5)

$$\begin{aligned}
V(n,0) &= C\\
\implies \int_0^{\pi}U(x,0)\sin nx dx &= C \qquad [Using (5.3)]\\
\implies C &= \left[\frac{-2x\cos nx}{n}\right]_0^{\pi} + \frac{2}{n}\int_0^{\pi}\cos nx dx\\
&= \left[\frac{-2\pi}{n}\cos n\pi - 0\right] + \frac{2}{n^2}\left[\sin nx\right]_0^{\pi}\\
\implies C &= \frac{-2\pi}{n}\cos n\pi. \qquad (5.6)
\end{aligned}$$

Putting C *in* (5.5)

$$V(n,t) = \frac{-2\pi}{n}\cos n\pi e^{-n^2 t} \qquad (5.7)$$

Now taking inverse sine transform

$$\begin{aligned}
U(x,t) &= \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{-2\pi}{n}\cos n\pi e^{-n^2 t}\sin nx\right)\\
U(x,t) &= 4\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{n}e^{-n^2 t}\sin nx\right)
\end{aligned}$$

Problem 5.2 Use Fourier transform to solve

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}, \qquad 0 < x < 6; t > 0.$$
$$U(0,t) = U(6,t) = 0, \qquad t > 0,$$
$$U(x,0) = \begin{cases} 1 & 0 < x < 3\\ 0 & 3 < x < 6.\end{cases}$$

Solution *Given,*

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \qquad (5.8)$$

Taking sine transform both side of (5.3)

$$\int_0^{6}\frac{\partial U}{\partial t}\sin\frac{n\pi x}{6}dx = \int_0^{\pi}\frac{\partial^2 U}{\partial x^2}\sin\frac{n\pi x}{6}dx \qquad (5.9)$$

Let

$$V = V(n,t) = \int_0^6 U(x,t)\sin\frac{n\pi x}{6}dx \tag{5.10}$$

Differentiating (5.10) *wrt t, we get*

$$\begin{aligned}
\frac{\partial V}{\partial t} &= = \int_0^6 \frac{\partial U}{\partial t}\sin\frac{n\pi x}{6}dx \\
&= \int_0^6 \frac{\partial^2 U}{\partial x^2}\sin\frac{n\pi x}{6}dx \qquad [Using(5.9)] \\
&= \left[\frac{\partial U}{\partial x}\sin nx\right]_0^6 - \frac{n\pi}{6}\int_0^6 \frac{\partial U}{\partial x}\cos\frac{n\pi x}{6}dx \\
&= 0 - \frac{n\pi}{6}\left[U(x,t)\cos\frac{n\pi x}{6}\right]_0^6 - \frac{n^2\pi^2}{36}\int_0^\pi U(x,t)\sin\frac{n\pi x}{6}dx \\
&= -\frac{n^2\pi^2}{36}V \qquad [Using(5.10)] \qquad (5.11) \\
\implies \frac{dV}{dt} &= -\frac{n^2\pi^2}{36}V \\
\implies \frac{dV}{V} &= -\frac{n^2\pi^2}{36}dt \\
\implies \ln V &= -\frac{n^2\pi^2}{36}t + \ln C \qquad [Integrating] \\
\implies V &= Ce^{-\frac{n^2\pi^2}{36}t} \qquad [Integrating] \qquad (5.12)
\end{aligned}$$

When $t = 0$ *then from* (5.12)

$$\begin{aligned}
V(n,0) &= C \\
\implies \int_0^6 U(x,0)\sin\frac{n\pi x}{6}dx &= C \qquad [Using(5.10)] \\
\implies C &= \int_0^3 U(x,0)\sin\frac{n\pi x}{6}dx + \int_3^6 U(x,0)\sin\frac{n\pi x}{6}dx \\
&= \int_0^3 \sin\frac{n\pi x}{6}dx + 0 \\
&= -\left[\cos\frac{n\pi x}{n}\right]_0^3 = 6\frac{1-\cos(n\pi/2)}{n\pi} \qquad (5.13)
\end{aligned}$$

Putting C *in* (5.12)

$$V(n,t) = 6\frac{1-\cos(\frac{n\pi}{2})}{n\pi}e^{-\frac{n^2\pi^2 t}{36}} \tag{5.14}$$

Now taking inverse sine transform

$$\begin{aligned}
U(x,t) &= \frac{2}{6}\sum_{n=1}^{\infty} 6\frac{1-\cos(\frac{n\pi}{2})}{n\pi}e^{-\frac{n^2\pi^2 t}{36}}\sin\frac{n\pi x}{6} \\
U(x,t) &= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos(\frac{n\pi}{2})}{n}e^{-\frac{n^2\pi^2 t}{36}}\sin\frac{n\pi x}{6}
\end{aligned}$$

Chapter 5 Exercise

1.

$$f_c\left\{\frac{\partial U}{\partial x}\right\} =?$$

2.
$$f_s\left\{\frac{\partial U}{\partial x}\right\}=?$$

3.
$$f_s\left\{\frac{\partial^2 U}{\partial x^2}\right\}=?$$

4.
$$f_c\left\{\frac{\partial^2 U}{\partial x^2}\right\}=?$$

5. By Fourier transform solve
$$\begin{aligned}\frac{\partial U}{\partial t}=\frac{\partial^2 U}{\partial x^2}, &\qquad 0<x<\pi, t>0\\ U(0,t)=U(\pi,t)=0, &\qquad t>0,\\ U(x,0)=2x, &\qquad 0<x<\pi.\end{aligned}$$

6. Use Fourier transform to solve
$$\begin{aligned}\frac{\partial U}{\partial t}=\frac{\partial^2 U}{\partial x^2}, &\qquad 0<x<6; t>0.\\ U(0,t)=U(6,t)=0, &\qquad t>0,\\ U(x,0)=\begin{cases}1 & 0<x<3\\ 0 & 3<x<6.\end{cases}\end{aligned}$$

Chapter 6 Frequency Distributions

Introduction

- *Population & Sample*
- *Frequency Distribution*

6.1 Population & Sample

Definition 6.1

A population *is the entire collection of all observation of interest to investigate and a* sample *is a representative portion of the population which is selected for study.* ♣

6.2 Frequency Distribution

Definition 6.2

Data are divided in to several classes, or categories, and determine the number of individuals belonging to each class, called the class frequency. ♣

Definition 6.3

A tabular arrangement of data by classes together with the corresponding class frequency is called a frequency distribution, *or* frequency table. ♣

Definition 6.4

A symbol $(a - b)$, *where* $a < b$ *defines a class is called a* class interval, *and the end numbers,* a, *and* b *are called lower and upper class limits respectively.* ♣

Definition 6.5

The size, or width, of a class interval is the difference between the lower and upper class boundaries and is also referred to as the class width, class size, *or* class length. ♣

Definition 6.6

The class mark, *is the mid point the class interval and is obtained by taking average of the corresponding class limits.* ♣

6.2.1 General Rules for Forming Frequency Distributions

1. Determine the largest and smallest numbers in the raw data and thus find the range (the difference between largest and smallest numbers).
2. Divide the range into a convenient number of class intervals having the same size. If this is not feasible, use class intervals of different sizes. The number of class intervals is usually between 5 and 20, depending on the data. Class intervals are also chosen so that the class marks (or midpoints) coincide with the actually

observed data. This tends to lessen the so-called grouping error involved in further mathematical analysis. However, the class boundaries should not coincide with the actually observed data.

3. Determine the number of observations falling into each class interval; that is, find the class frequencies. This is best done by using a tally,or score sheet.

Problem 6.1 Prepare a frequency distribution from the following data:

33	32	47	55	21	50	27	12	68	49	40	17	44	62	24
42	33	38	45	26	44	33	48	52	30	50	37	38	45	48

Solution *Range is 68-12=56. If 5 class intervals are used, the class interval size is* $56/5 \equiv 11$*, if 20 class intervals are used, the class interval size is* $56/20 \equiv 3$*. One convenient choice for the class interval size is 5. Also, it is convenient to choose the class* $10, 15, 20, \ldots.$ *Thus the class intervals can be taken as* $8-12, 13-17, 18-22, \ldots.$ *With the choice the class boundaries are* $7.5, 12.5, 17.5, \ldots,$ *which do not coincide with the observed data.*

Data	Tally	Frequency
8-12		1
13-17		1
18-22		1
23-27		3
28-32		2
33-37		4
38-42		4
43-47		5
48-52		6
53-57		1
58-62		1
63-68		1
Total		30

Chapter 6 Exercise

1. Define population?
2. What do you mean by frequency?
3. Prepare a frequency distribution from the following data:

33	32	47	55	21	50	27	12	68	49	40	17	44	62	24
42	33	38	45	26	44	33	48	52	30	50	37	38	45	48

Chapter 7 Measures of Central Tendency and Dispersion

Introduction

- *Central Tendency*
- *Measures of Central Tendency*
- *Arithmetic Mean*
- *Geometric Mean*
- *Median*
- *Mode*
- *Measure of Dispersion*
- *Variation*

7.1 Central Tendency

Definition 7.1

Central tendency *is a typical value which is representative of the entire group of data.* ♣

7.2 Measures of Central Tendency

The following are the five measures of central tendency that are in common use:

1. Arithmetic Mean,
2. Median,
3. Mode,
4. Geometric Mean, and
5. Harmonic Mean.

7.3 Arithmetic Mean

Definition 7.2

The arithmetic mean, *or briefly the mean, of a set of N numbers $X_1, X_2, X_3, \ldots, X_N$ is denoted by $\bar{X}$ and is defined as*

$$\bar{X} = \frac{X_1 + X_2 + X_3 + \cdots + X_N}{N} = \frac{\sum_{i=1}^{N} X_i}{N} = \frac{\sum X}{N}$$

♣

If the numbers $X_1, X_2, X_3, \ldots, X_k$ occur $f_1, f_2, f_3, \ldots, f_k$ times, respectively, then the arithmetic mean is

$$\bar{X} = \frac{f_1X_1 + f_2X_2 + f_3X_3 + \cdots + X_k}{f_1 + f_2 + f_3 + \cdots + f_k} = \frac{\sum_{i=1}^{k} f_iX_i}{\sum_{i=1}^{k} f_i} = \frac{\sum fX}{N},$$

where $N = \sum f$ is the total frequency.

7.4 Median

Definition 7.3

The median of a set of N numbers arranged in order of magnitude is the middle value if N is odd, or the average of the middle two values if N is even. ♣

For grouped data, the median, obtained by interpolation, is given by

$$\text{Median} = L_1 + \left(\frac{\frac{N}{2} - (\sum f)_l}{f_{\text{median}}} \right) c$$

where L_1 = lower class boundary of the median class

N = number of items in the data

$(\sum f)_l$ = sum of the frequency of all classes lower than the median class

f_{median} = frequency of the median class

c = size of the median class interval.

7.5 Mode

Definition 7.4

The mode of a set of numbers is that value which occurs with the greatest frequency. The mode may not exist, and even if it exist it may not be unique. ♣

In the case of grouped data where a frequency curve has been constructed to fit the data, the mode will be the value (or values) of X corresponding to the maximum point (or points) on the curve. This value of X is sometimes denoted by $\hat{X}$. From a frequency distribution or histogram the mode can be obtained from the formula

$$\text{Mode} = L_1 + \left(\frac{\Delta_1}{\Delta_1 + \Delta_2} \right) c$$

where L_1 = lower class boundary of the modal class (i.e., the class containing the mode)

Δ_1 = excess of modal frequency over frequency of next-lower class

Δ_2 = excess of modal frequency over frequency of next-higher class

c = size of the modal class interval.

7.6 Geometric Mean

Definition 7.5

The geometric mean G of a set of N positive numbers $X_1, X_2, X_3, \ldots, X_N$ is the Nth root of the product of the numbers.

$$G = \sqrt[N]{X_1 X_2 X_3 \ldots X_N}$$

♣

Example 7.1 The geometric mean of the numbers 2, 4, and 8 is

$$G = \sqrt[3]{2 \cdot 4 \cdot 8} = 4.$$

Problem 7.1 Marks obtained by 10 students given below:

40, 30, 80, 70, 50, 20, 48, 95, 12, 18 compute the mean, and median.

Solution *Mean* $= \frac{\sum X}{N} = \frac{40+30+80+70+50+20+48+95+12+18}{10} = \frac{463}{10} = 46.3.$

The given marks can be sorted as follows:

12, 18, 20, 30, 40, 48, 50, 70, 80, 95.

Here, $N = 10$ *is even. So, Median* $= \frac{X_5+X_6}{2} = \frac{40+48}{2} = 44.$

7.7 Measure of Dispersion

Problem 7.2 The following table gives the height (in inches) of 100 students of class. Compute mean, mode, and median of the height. Also comment about the name of the distribution:

Height (inches)	60-62	62-64	64-66	66-68	68-70	70-72
No. of students	5	18	42	20	8	7

Solution *We have, Mean* $= \frac{\sum fX}{N} = \frac{6558}{100} = 65.58.$

Height (inches)	class Mark (X)	Frequency f	cf	fX
60-62	61	5	5	305
62-64	63	18	23	1134
64-66	65	42	65	2730
66-68	67	20	85	1340
68-70	69	8	93	552
70-72	71	7	100	497
	Total	100		6558

$$Median = L_1 + \left(\frac{\frac{N}{2} - (\sum f)_l}{f_{median}}\right) c = 64 + \left(\frac{\frac{100}{2} - 23}{42}\right) 2 = 65.29$$

$$Mode = L_1 + \left(\frac{\Delta_1}{\Delta_1 + \Delta_2}\right) c = 64 + \left(\frac{42 - 18}{42 - 18 + 42 - 20}\right) 2 = 65.04$$

Here, mean $>$ *median* $>$ *mode, nature of data is positive. So the distribution is positive skewed.*

7.8 Dispersion

7.8.1 Significance of Measuring Dispersion

Measures of dispersion are needed for four basic significance

1. **To determine the reliability of an average:** Measure of dispersion point out as to how for an average is representative of the entire data. On the other hand, when variation is large the average is not so typical, and unless the sample is very large, the average may be quite unreliable.
2. **To serve as a basis for the control of the variability:** Another purpose of measuring variation is to determine nature and cause of variation in order to control the variation itself. Thus measurement of variation is basic to the control of cause of variation.
3. **To compare two or more series with regard to the variability:** Measures of variation enable comparison to be made of two or more series with regard to their variability.
4. **To facilitate the the use of other statistical measures:** Many powerful analytical tools in statistics such as correlation analysis, the testing of hypothesis, the analysis of fluctuations, techniques of production control, cost control, etc. are based on measure of variation of one kind or another.

7.9 Variation

Definition 7.6

The standard deviation *of a set of N numbers $X_1, X_2, X_3, \ldots, X_N$ is denoted by s and is defined by*

$$s = \sqrt{\frac{\sum_{N}^{i=1} (X_i - \bar{X})^2}{N}}$$

♣

Definition 7.7

The variance *of a set of data is defined as the square of the standard deviation and is denoted by s^2, mathematically can be written as*

$$v = s^2 = \frac{\sum_{N}^{i=1} (X_i - \bar{X})^2}{N}$$

♣

Chapter 7 Exercise

1. Define central tendency.
2. Find the geometric mean of the series 2, 4, and 8.
3. Marks obtained by 10 students given below:
   40, 30, 80, 70, 50, 20, 48, 95, 12, 18 compute the mean, and median.
4. Mention the significance of measuring dispersion.
5. The following table gives the height (in inches) of 100 students of class. Compute mean, mode, and median of the height. Also comment about the name of the distribution:

Height (inches)	60-62	62-64	64-66	66-68	68-70	70-72
No. of students	5	18	42	20	8	7

Chapter 8 Skewness, & Kurtosis

Introduction

- *Skewness*
- *Kurtosis*

8.1 Skewness

Definition 8.1

Skewness *is the degree of asymmetry, or departure from symmetry, of a distribution.* ♣

$$\text{Skewness} = \frac{\text{mean} - \text{mode}}{\text{standard deviation}} = \frac{\text{mean} - \text{mode}}{s}$$

8.2 Kurtosis

Definition 8.2

Kurtosis *is the degree of peakedness of a distribution, usually taken relative to a normal distribution.* ♣

8.2.1 Distinguish between Skewness and Kurtosis

There are some difference between skewness and kurtosis are as follows:

Subject	Skewness	Kurtosis
Definition	Skewness is the degree of asymmetry, or departure from symmetry, of a distribution.	Kurtosis is the degree of peakedness of a distribution, usually taken relative to a normal distribution.
Measurement	With the help of skewness the shape of distribution can be measured.	With
Result	Positive, negative or zero.	Lepto, Meso, or platy kurtic.
For normal distribution	Skeness is zero	Kurtosis is meso-Kurtic
Formula	$\text{Sk} = \frac{\text{Mean}-\text{Mode}}{s}$	$\beta_2 = \frac{m_4}{s^4}$

Chapter 8 Exercise

1. Distinguish between Skewness and kurtosis.

Chapter 9 Correlation Analysis

Introduction

❏ *Correlation*

9.1 Correlation

Problem 9.1 The data related to capital and profit of five shops are given:

Capital (in lac TK.): x	5	10	15	20	25
Profit (in lac TK.): y	3	4	8	12	18

1. Draw a scatter diagram.
2. Compute the co-efficient of correlation and interpret its value.

Solution *We know, Co-efficient of correlation:*

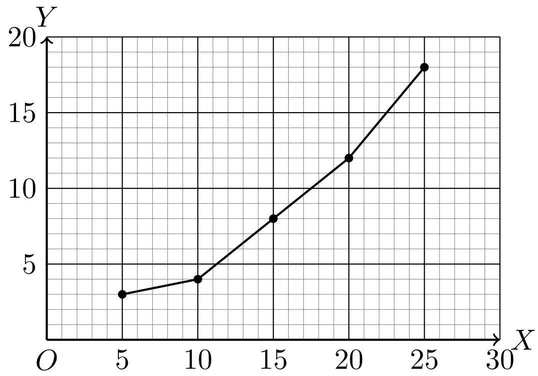

Figure 9.1: Scatter diagram

Capital (x)	x^2	Profit (y)	y^2	xy
5	25	3	9	15
10	100	4	16	40
15	225	8	64	120
20	400	12	144	240
25	625	18	324	450
$\sum x = 75$	$\sum x^2 = 1375$	$\sum y = 45$	$\sum y^2 = 557$	$\sum xy = 865$

$$\begin{aligned} r &= \frac{\sum xy - \frac{\sum x \sum y}{N}}{\sqrt{\left(\sum x^2 - \frac{(\sum x)^2}{N}\right)\left(\sum y^2 - \frac{(\sum y)^2}{N}\right)}} \\ &= \frac{865 - \frac{75 \cdot 45}{5}}{\sqrt{\left(1375 - \frac{75^2}{5}\right)\left(557 - \frac{45^2}{5}\right)}} \\ &= \frac{865 - 675}{\sqrt{(1375 - 1125)(557 - 405)}} = \frac{190}{\sqrt{250 \cdot 152}} = 0.97 \end{aligned} \tag{9.1}$$

Probable Error (P.E) =

$$0.6745\left(\frac{1-r^2}{\sqrt{N}}\right) = 0.6745\left(\frac{1-0.97^2}{\sqrt{5}}\right) = 0.6745 \cdot 0.0268 = 0.018 \tag{9.2}$$

6 times of P.E = 0.108

Since the value $r > 6P.E$, therefore the correlation is significant.

9.1.1 Significance of Measuring Correlation

Correlation is an important tool that is used in analyzing, measuring and interpreting the relationship between two or more variables. The significance of measuring correlation are stated below:

1. **Nature of relationship:** Through correlation we can measure the nature of relationship between variables. If we can determine the relationship, we shall be able to take proper decision. If the value of $'r'$ is positive, we can understand that increase of one variable causes increase of another variable. On the other hand, if it is negative negative, increases of one variable causes decrease of another variable.
2. **Strength of relationship:** By measuring correlation, we can know the strength of the relationship between variables. If the values of r 1 or near 1, we consider that relationship is very strong. On the other hand, if the value of r is zero or near zero we say that the relationship is weak.
3. **Effect:** From the coefficient of determination, we come to know, what portion of the variation of the dependent variable is affected by the independent variable. If the value of r^2 is equal to 0.64, we understand that 64% of dependent variable is affected by the independent variable.
4. **Relationship among economic variables** Coefficient of correlation help help to analyze the relationship among the economic variables such as demand and supply, advertisement and sales, cost and revenue and so on.

   Construction regression line: Coefficient of relation is also used in determining regression lines.
5. **Interpretation of relationship:** By measuring the correlation we can interpret the relationship between variables.

Chapter 9 Exercise

1. Write down the significance of measuring correlation.
2. The data related to capital and profit of five shops are given:

Capital (in lac TK.): x	5	10	15	20	25
Profit (in lac TK.): y	3	4	8	12	18

   (a). Draw a scatter diagram.

   (b). Compute the co-efficient of correlation and interpret its value.

Chapter 10 Regression Analysis

Problem 10.1 From the following regression equations calculate the coefficient of correlation:

$$x = 5.28 + 0.59y$$
$$y = 1.34x + -5.40$$

Solution *We know that the regression equation of x on y is $x = a_1 + b_1 y$, which provide $b_1 = 0.59$. Again the regression equation of y on x is $y = a_2 + b_2 y$, which provide $b_2 = 1.34$.*

We know, coefficient of correlation $r = \sqrt{b_1 b_2} = \sqrt{0.59 \cdot 1.34} = 0.889$.

Chapter 10 Exercise

1. From the following regression equations calculate the coefficient of correlation:

$$x = 5.28 + 0.59y$$
$$y = 1.34x + -5.40$$

Chapter 11 Elementary Probability Theory

Introduction

- *Probability*
- *Conditional Probability*

11.1 probability

Definition 11.1

Let an event E can happen in n ways out of total N possible equally likely ways. Then the probability *of occurrence of the event (called its* success*) is denoted by*

$$p = P\{E\} = \frac{n}{N}.$$

The probability *of nonoccurence of the event (called its* failure*) is denoted by*

$$p = P\{\bar{E}\} = \frac{N-n}{N} = 1 - \frac{n}{N}.$$

♣

11.2 Conditional Probability

Definition 11.2

If E_1 and E_2 are two events, the probability that E_2 occurs given that E_1 has occurred is denoted by $P\{E_2|E_1\}$, and is called the conditional probability *of E_2 given that E_1 has occurred.* ♣

Definition 11.3

If the occurrence of the event E_1 does not effect the probability of occurrence of the event E_2, then $P\{E_2|E_1\} = P\{E_2\}$ and we say that E_1 and E_2 are independent events; otherwise, they are dependent events. ♣

11.2.1 Compound Event

If we denote E_1E_2 the event that "both E_1 and E_2 occur," sometimes called a compound event, then

$$P\{E_1E_2\} = P\{E_1\}P\{E_2|E_1\}.$$

For independent events

$$P\{E_1E_2\} = P\{E_1\}P\{E_2\}.$$

11.2.2 Mutually Exclusive Events

Two or more events are called mutually exclusive if the occurrence of anyone of them excludes the occurrence of the others. Thus if E_1 and $E2$ are mutually exclusive events, then $P\{E_1E_2\} = 0$. If $E_1 + E_2$ denotes the event that "either E_1 or E_2 or both occur," then

$$P\{E_1 + E_2\} = P\{E_1\} + P\{E_2\} - P\{E_1E_2\}.$$

In particular,

$$P\{E_1 + E_2\} = P\{E_1\} + P\{E_2\}$$

or mutually exclusive events.

Example 11.1 There are 5 red and 4 white balls in a bag. One ball is drawn from the bag, What is the probability that is either red or white.

Solution *There are total $5 + 4 = 9$ balls. Let $R =$ event "red ball is drawn" and $W =$ event "white ball is drawn".*

$$P\{R + W\} = P\{R\} + P\{W\} = \frac{5}{9} + \frac{4}{9} = 1.$$

Problem 11.1 There are 5 white and 7 red balls in a bag. Two balls are drawn such that a ball is drawn and replaced. What is the probability that a white ball and a red ball are drawn in that order? What would be the probability if the balls are drawn were not put back in to the bag.

Solution *There are total $5 + 7 = 12$ balls. Let $W =$ event "white" on the first draw, and $R =$ event "red" on the second draw.*

(a). If each ball is replaced, then W, and R are independent events and

$$P\{WR\} = P\{W\}P\{R\} = \frac{5}{12}\frac{7}{12} = \frac{35}{144}.$$

(b). If each ball is not put back, then W, and R, are dependent events and

$$P\{WR\} = P\{W\}P\{R|W\} = \frac{5}{12}\frac{7}{11} = \frac{35}{132}.$$

Problem 11.2 Three balls are drawn successively from the box of containing 6 red balls, 4 white balls, and 5 blue balls. Find the probability that they are drawn in the order red, white, and blue if each ball is (a) replaced and (b) not replaced.

Solution *There are total $6 + 4 + 5 = 15$ balls. Let $R =$ event "red" on the first draw, $W =$ event "white" on the second draw, and $B =$ event "blue" on the third draw.*

(a). If each ball is replaced, then R, W, and B are independent events and

$$P\{RWB\} = P\{R\}P\{W\}P\{B\} = \frac{6}{15}\frac{4}{15}\frac{5}{15} = \frac{8}{225}.$$

(b). If each ball is not replaced, then R, W, and B are dependent events and

$$P\{RWB\} = P\{R\}P\{W|R\}P\{B|WR\} = \frac{6}{15}\frac{4}{14}\frac{5}{13} = \frac{4}{91}.$$

Problem 11.3 A fair die is tossed twice. Find the probability of getting a 4, 5, or 6 on the first toss and a 1, 2, 3, or 4 on the second toss.

Solution *Let $E_1 =$ event "4, 5, or 6" on the first toss, and let $E_2 =$ event "1, 2, 3, or 4" on the second toss. Each of the six ways in which the die can fall on the first toss can be associated with each of the six ways in which it can fall on the second toss, a total of $6 \cdot 6 = 36$ ways, all equally likely. Each of the three ways in which E_1 can occur can be associated with each of the four ways in which $E2$ can occur, to give $3 \cdot 4 = 12$ ways in which both E_1 and $E2$, or E_1E_2 occur. Thus $P(E_1E_2) = 12/36 = 1/3$.*

Problem 11.4 A and B play 12 games of chess, of which 6 are won by A, 4 are won by B, and 2 end in draw. They agree to play a match consisting of 3 games. Find the probability that

(a). A wins all 3 games,

(b). 2 games end in a draw,

(c). A and B win alternately, and

(d). B wins at least 1 game.

Solution *Let A_1, A_2, and A_3 denote the events "A wins" in the first, second, and third games,respectively; let*

B_1, B_2, and B_3 denote the events "B wins" in the first, second, and third games, respectively; and let, D_1, D_2, and D_3 denote the events "there is a draw" in the first, second, and third games, respectively.

On the basis of their past experience(empirical probability), we shall assume that $P(A)$ = P(A wins anyone games) $= \frac{6}{12} = \frac{1}{2}$, that $P(B)$ = P(B wins anyone games $= \frac{4}{12} = \frac{1}{3}$, and that $P(D)$ = P(anyone games ends in a draw$= \frac{2}{12} = \frac{1}{6}$.

(a). P(A wins all games) $= P\{A_1A_2A_3\} = P(A_1)P(A_2)P(A_3) = \frac{1}{2}\frac{1}{2}\frac{1}{2} = \frac{1}{8}$.

(b). P(2 games end in a draw)

$$\begin{aligned} &= P\{D_1D_2\bar{D}_3\} + P\{D_1\bar{D}_2D_3\} + P\{\bar{D}_1D_2D_3\} \\ &= P(D_1)P(D_2)P(\bar{D}_3) + P(D_1)P(\bar{D}_2)P(D_3) + P(\bar{D}_1)P(D_2)P(D_3) \\ &= \frac{1}{6}\frac{1}{6}\frac{5}{6} + \frac{1}{6}\frac{5}{6}\frac{1}{6} + \frac{5}{6}\frac{1}{6}\frac{1}{6} = \frac{15}{216} = \frac{5}{72}. \end{aligned}$$

(c). P(A and B win alternately)

$$\begin{aligned} &= P\{A_1B_2A_3 + B_1A_2B_3\} = P\{A_1B_2A_3\} + P\{B_1A_2B_3\} \\ &= P(A_1)P(B_2)P(A_3) + P(B_1)P(A_2)P(B_3) \\ &= \frac{1}{2}\frac{1}{3}\frac{1}{2} + \frac{1}{3}\frac{1}{2}\frac{1}{3} = \frac{1}{12} + \frac{1}{18} = \frac{5}{36} \end{aligned}$$

(d). P(B wins at least 1 game) $= 1-$ *P(B wins no game)* $=$

$$1 - P(\bar{B}_1\bar{B}_2\bar{B}_3) = 1 - P(\bar{B}_1)P(\bar{B}_2)P(\bar{B}_3) = 1 - \frac{2}{3}\frac{2}{3}\frac{2}{3} = \frac{19}{27}.$$

Chapter 11 Exercise

1. What is conditional probability?
2. There are 5 red and 4 white balls in a bag. One ball is drawn from the bag, What is the probability that is either red or white.
3. There are 5 white and 7 red balls in a bag. Two balls are drawn such that a ball is drawn and replaced. What is the probability that a white ball and a red ball are drawn in that order? What would be the probability if the balls are drawn were not put back in to the bag.
4. Three balls are drawn successively from the box of containing 6 red balls, 4 white balls, and 5 blue balls. Find the probability that they are drawn in the order red, white, and blue if each ball is (a) replaced and (b) not replaced.
5. A fair die is tossed twice. Find the probability of getting a 4, 5, or 6 on the first toss and a 1, 2, 3, or 4 on the second toss.
6. A and B play 12 games of chess, of which 6 are won by B, and 2 end in draw. They agree to play a match consisting of 3 games. Find the probability that
   (a). A wins all 3 games,
   (b). 2 games end in a draw,
   (c). A and B win alternately, and
   (d). B wins at least 1 game.

Chapter 12 Test of Hypothesis

Introduction

- *Null hypothesis*
- *Steps of Testing*

12.1 Null Hypothesis

Definition 12.1
The hypothesis about a population parameter we wish to test is called a null hypothesis*. For every null hypothesis there is an alternative hypothesis.* ♣

12.2 Steps of Testing

The process of reaching decision about the population by taking and analyzing sample from the population is called the testing of hypothesis. That means decision about the population is taken by hypothesis testing. Several steps are followed in hypothesis testing:

1. **Set up a Hypothesis:** The first step in hypothesis testing is to establish the hypothesis to be tested. The hypothesis are normally referred to as
   (a). null hypothesis denoted by H_0, and
   (b). Alternative hypothesis denoted by H_1.
   Both null and alternative hypothesis must be stated in statistic terms using populations parameters.
2. **Choose the level of significance:** Having set up a hypothesis, the next step is to select a suitable level of significance.
3. Determine the appropriate statistical technique and corresponding test static to use.
4. Determine the critical region. Set up the critical values that divide the rejection and non rejection regions.
5. Collect the data and compute the sample values of the appropriate test statistic.
6. Determine whether the test statistic has fallen into the rejection or non rejection region. The computed value of the test statistic is compared with critical values for the appropriate sampling distribution to determine whether it falls into the rejection or non rejection region.
7. Make statistical decision. If the test statistic falls into the non rejection region, the hypothesis H_0 can not be rejected. If the test statistic falls into the rejection region, the null hypothesis is rejected.
8. Express the statistical decision is the context of the problem.

Problem 12.1 A random sample of 200 tins of coconut oil gave an average weight of 4.95 kg with a standard deviation of 0.21 kg. At 1% level of significance, can we say that net weight is 5 kg per tin?

Solution *We know, Standard Error (S.E)* $= \frac{s}{\sqrt{N}} = \frac{0.21}{\sqrt{200}} = 0.01485$.

Now $|Z| = \frac{5-4.95}{0.01485} = 3.367$.

Here, $Z_{cal} = 3.367 > 2.58$. *Therefore the difference is not significant, and we can not say that net weight of a tin is 5 kg per tin.*

Chapter 12 Exercise

1. What is null hypothesis?
2. Discuss the steps of hypothesis testing.
3. A random sample of 200 tins of coconut oil gave an average weight of 4.95 kg with a standard deviation of 0.21 kg. At 1% level of significance, can we say that net weight is 5 kg per tin?
4. The standard deviation of the lifetimes of 200 electric bulbs is 100 hours. Find the
   (a). 95%, and
   (b). 99% confidence limits for the standard deviation of such electric bulbs.