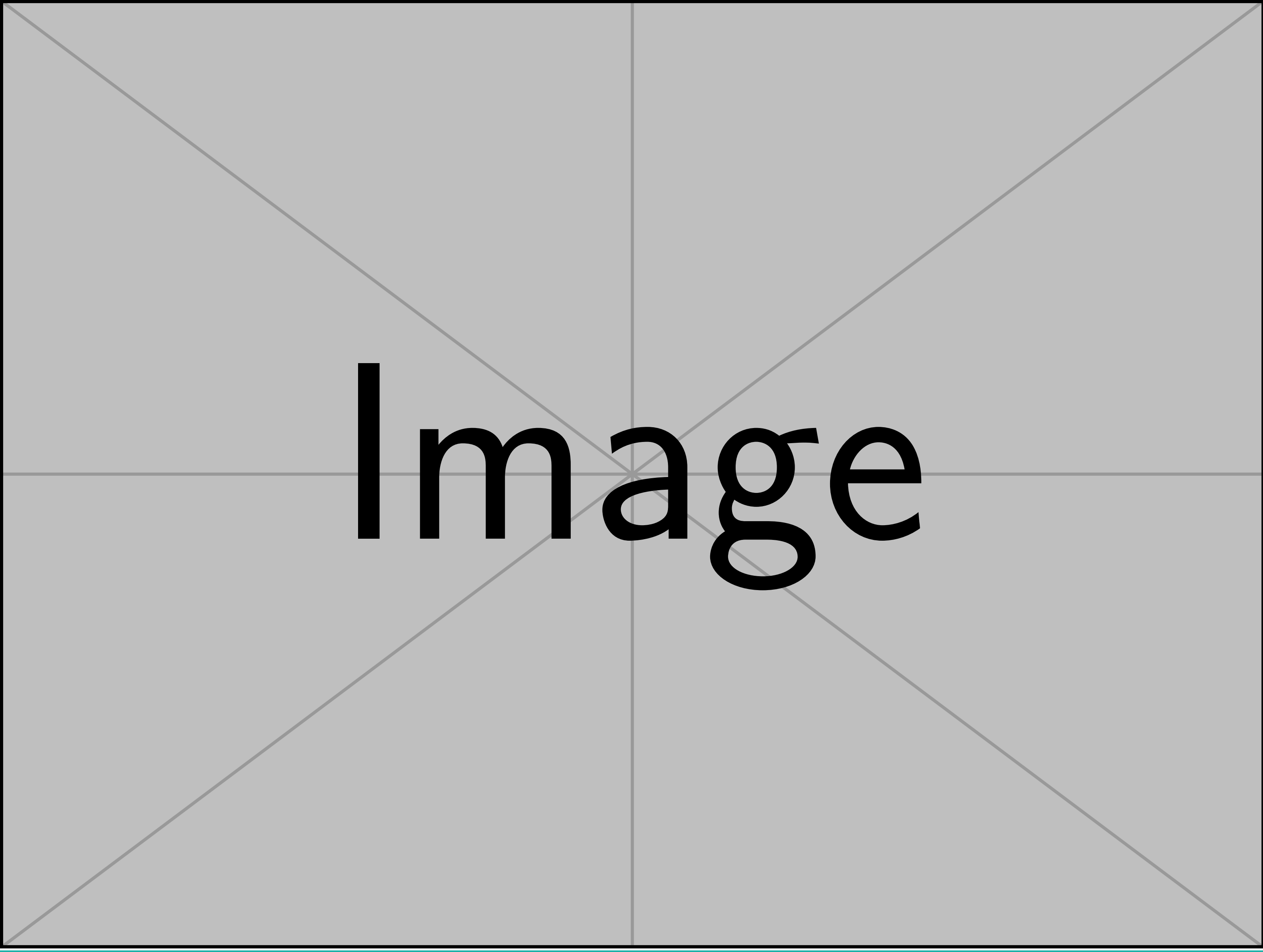

Notes on Mathematics IV

Matrices, Geometry & Harmonic Function

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Victory won't come to us unless we go to it.

Contents

Chapter 1 Systems of Linear Equations

Introduction

- Introduction to Systems of Linear Equations
- Solution of a Non-homogeneous System of Linear Equations
- Solution of a Homogeneous System of Linear Equations

1.1 Introduction to Systems of Linear Equations

Let the two equations be

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2.$$

If we interpret x, y as coordinate in xy-plane, then each of the above two linear equations represents a straight line and (α, β) be a solution of the above two equations if and only if the P with the coordinates α, β lies on both lines. Hence there are three possible cases:

- No solution if the lines are parallel.
- Precisely one solution if they intersect.
- Infinitely many solutions if they coincide.

These cases are illustrated by the following examples:

Example 1.1 The linear system

$$x + y = 1$$
$$x + y = 3$$

has no solution. Since the lines represented by two lines are parallel **Figure** 1.1.

Example 1.2 The linear system

$$x + y = 3$$
$$x - y = 1$$

has only one solution. Since the lines represented by two lines intersect at $(2, 1)$ **Figure** 1.2.

Example 1.3 The linear system

$$x + y = 3$$
$$2x + 2y = 6$$

has infinitely many solutions. Since the lines represented by two linear equations coincide **Figure** 1.3.

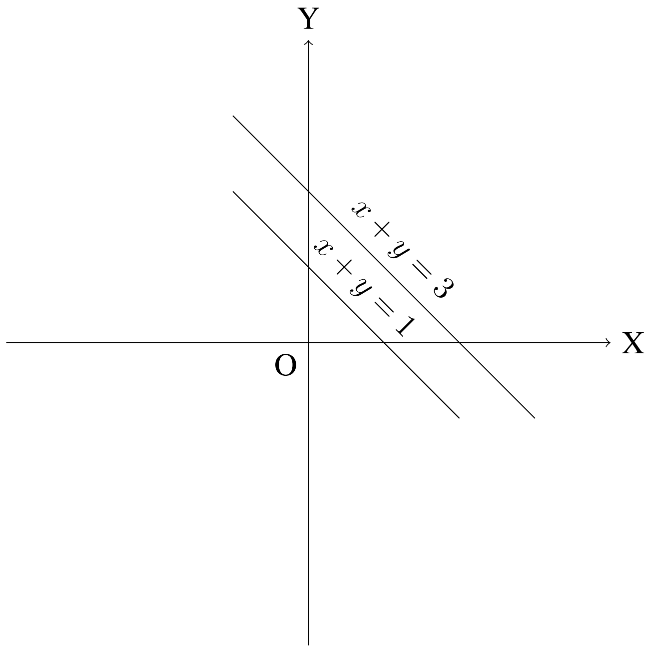


Figure 1.1: Example 1: Two lines are parallel.

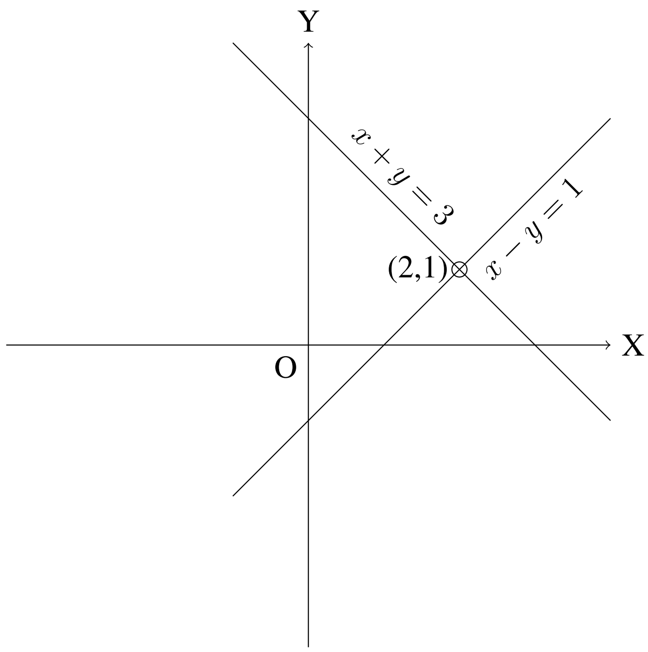


Figure 1.2: Example 2: Two lines are intersect at $(2, 1)$.
X

1.2 Solution of a Non-homogeneous System of Linear Equations

1.3 Solution of a Homogeneous System of Linear Equations

Chapter 1 Exercise

1. Define the following:

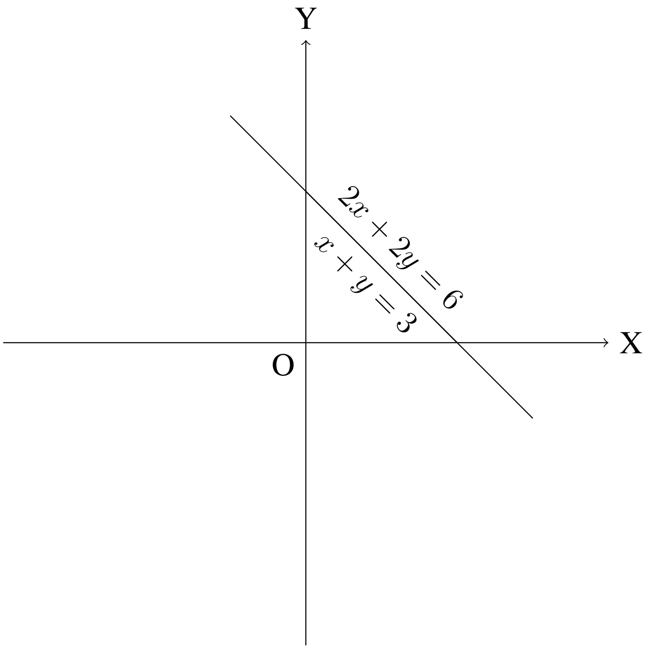


Figure 1.3: Example 3: Two lines coincide.

(a). Hermitian matrix

2. Determine the values of λ so that the following system has (i) unique solution, (ii) more than one solution, and (iii) no solution.

$$x + y - z = 1$$

$$2x + 3y + \lambda z = 3$$

$$x + \lambda y + 3z = 2$$

and hence find all the solutions.

Chapter 2 Matrix Algebra

Introduction

- Introduction to Matrix
- Different Types of Matrices
- Algebra of Matrices
- Adjoin of Matrices
- Inverse of Matrices

2.1 Introduction to Matrix

Definition 2.1

A matrix *(over the field* $\mathbb{R} or \mathbb{C}$*) is a rectangular array of numbers (real or complex) enclosed by pair of brackets i.e.*

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & \dots & a_{mn} \end{bmatrix},$$

the numbers a_{ij} *in the matrix are called the* entries *or the* element *of the matrix. The matrix of* m *rows and* n *columns is said to be of* order *"*m *by* n*", or* $m \times n$*.* ♣

2.2 Different Types of Matrices

Definition 2.2

A square matrix whose elements $a_{ij} = 0$ *for* $i > j$ $(i \geq j)$ *is called an* (strictly) upper triangular matrix. ♣

Definition 2.3

A square matrix whose elements $a_{ij} = 0$ *for* $i < j$ $(i \leq j)$*is called a* (strictly) lower triangular matrix. ♣

Definition 2.4

A square matrix A *is said to be an* idempotent matrix *if* $A^2 = A$*.* ♣

2.3 Transpose of Matrices

Definition 2.5

If $A = (a_{ij})_{m\times n}$ is a matrix over the field $\mathbb{R}$, then the matrix $A^T = (a_{ji})_{n\times m}$ obtained from the matrix A by writing its rows as column and columns as rows is called the transpose *of A.* ♣

Example 2.1 Find the transpose of the matrix

$$\begin{pmatrix} 2 & 1 & -1 \\ 3 & 2 & 3 \\ 1 & 4 & 2 \end{pmatrix}.$$

Solution *Transpose of the given matrix is*

$$\begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 4 \\ -1 & 3 & 2 \end{pmatrix}.$$

Theorem 2.1

Prove that

$$(AB)^T = B^T A^T,$$

where A and B are matrices. ♡

Proof Let $A = (a_{ij})_{m\times n}$ and $B = (b_{jk})_{n\times p}$. Then

$$A^T = ((a_{ij})_{m\times n})^T = (a_{ji})_{n\times m}$$

and

$$B^T = ((b_{jk})_{n\times p})^T = (b_{kj})_{p\times n}$$

Thus AB is a $m \times p$ matrix so that $(AB)^T$ is a $p \times m$ matrix. Also $B^T A^T$ is a $p \times m$ matrix. Therefore, $(AB)^T$ and $B^T A^T$ have same dimensions.

Now let $AB = (c_{ik})_{m\times p}$, where

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$$

That is (k, i)th element of $(AB)^T$ is

$$\sum_{j=1}^{n} a_{ij} b_{jk} = \sum_{j=1}^{n} a_{ji}^T b_{kj}^T = \sum_{j=1}^{n} b_{kj}^T a_{ji}^T$$

(k, i)th element of $B^T A^T$. That is, $(AB)^T = B^T A^T$.

2.4 Conjugate Transpose of Matrices

If $a = x + iy$ then $\bar{a} = x - iy$.

Definition 2.6

If $A = (a_{ij})_{m\times n}$ is a matrix over the field $\mathbb{C}$, then the matrix $\bar{A} = (\bar{a}_{ij})_{m\times n}$ is called the conjugate *of A.* ♣

Definition 2.7

The conjugate of the transpose of a complex matrix $A = a_{i\times j}$ is said to be conjugate transpose *of A and is denoted by A^*, that is*

$$A^* = (\bar{A})^T = \bar{A}^T = (\bar{a}_{j\times i}).$$

♣

Definition 2.8

If $A = (a_{ij})_{n\times n}$ is a square matrix over the complex field and $A^* = \bar{A}^T = A$ i.e. $(a_{ij})_{n\times n} = (\bar{a}_{ji})_{n\times n}$. ♣

Theorem 2.2

If A is a square matrix over the complex field then A can be expressed uniquely as the sum of a Hermitian matrix, and a skew-Hermitian matrix. ♡

Proof Let A^* be the conjugate transpose of A. Then we can write

$$A = \frac{1}{2}(A + A^*) + \frac{1}{2}(A - A^*) = P + Q. \tag{2.1}$$

where $P = \frac{1}{2}(A + A^*)$ and $Q = \frac{1}{2}(A - A^*)$. Now

$$P^* = \frac{1}{2}(A + A^*)^* = \frac{1}{2}(A^* + (A^*)^*) = \frac{1}{2}(A^* + A) = P$$

and

$$Q^* = \frac{1}{2}(A - A^*)^* = \frac{1}{2}(A^* - (A^*)^*) = \frac{1}{2}(A^* - A) = -Q$$

Thus P is Hermitian and Q is skew-Hermitian matrix. Hence, from (2.1), if A is a square matrix over the complex field then A can be expressed as the sum of a Hermitian matrix, and a skew-Hermitian matrix.

To prove the uniqueness of (2.1), let

$$A = R + S \tag{2.2}$$

, where $R \neq P$ is Hermitian and $S \neq Q$ is skew-Hermitian. Now

$$A^* = (R + S)^* = R^* + S^* = R - S \tag{2.3}$$

Adding (2.2) and (2.3)

$$R = \frac{1}{2}(A + A^*) = P$$

and subtracting (2.3) from (2.2)

$$S = \frac{1}{2}(A - A^*) = Q.$$

Which establish the uniqueness of (2.1). Hence, the theorem is proved.

2.5 Inverse of Matrices

Problem 2.1 Find the inverse of the following matrix $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$.

Solution *The inverse of the given matrix is* $\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$.

Problem 2.2 Find the inverse of the following matrix

$$\begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix},$$

Solution *Given*

$$A = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix},$$

then

$$D = |A| = \begin{vmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{vmatrix} = -1(5+0) - 2(10-0) - 3(-4-4) = -5 - 20 + 24 = -1 \neq 0.$$

So, A is non-singular and hence A^{-1} exists.

$$\begin{aligned}
\textit{Co-factor of } -1 &= A_{11} = \begin{vmatrix} 1 & 0 \\ -2 & 5 \end{vmatrix} = 5 \\
\textit{Co-factor of } 2 &= A_{12} = (-1)\begin{vmatrix} 2 & 0 \\ 4 & 5 \end{vmatrix} = -10 \\
\textit{Co-factor of } -3 &= A_{13} = \begin{vmatrix} 2 & 1 \\ 4 & -2 \end{vmatrix} = -8 \\
\textit{Co-factor of } 2 &= A_{21} = (-1)\begin{vmatrix} 2 & -3 \\ -2 & 5 \end{vmatrix} = -4 \\
\textit{Co-factor of } 1 &= A_{22} = \begin{vmatrix} -1 & -3 \\ 4 & 5 \end{vmatrix} = 7 \\
\textit{Co-factor of } 0 &= A_{23} = (-1)\begin{vmatrix} -1 & 2 \\ 4 & -2 \end{vmatrix} = 6 \\
\textit{Co-factor of } 4 &= A_{31} = \begin{vmatrix} 2 & -3 \\ 1 & 0 \end{vmatrix} = 3 \\
\textit{Co-factor of } -2 &= A_{32} = (-1)\begin{vmatrix} -1 & -3 \\ 2 & 0 \end{vmatrix} = -6 \\
\textit{Co-factor of } 5 &= A_{33} = \begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} = -5
\end{aligned}$$

$$Adj\ A = \begin{bmatrix} 5 & -10 & -8 \\ -4 & 7 & 6 \\ 3 & -6 & -5 \end{bmatrix}^T = \begin{bmatrix} 5 & -4 & 3 \\ -10 & 7 & 3 \\ -8 & 7 & -5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{D} Adj\ A = \frac{1}{-1} \begin{bmatrix} 5 & -4 & 3 \\ -10 & 7 & 3 \\ -8 & 7 & -5 \end{bmatrix} = \begin{bmatrix} -5 & 4 & -3 \\ 10 & -7 & -3 \\ 8 & -7 & 5 \end{bmatrix}.$$

Problem 2.3 Given that $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$, and $B = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$, verify that $(AB)^{-1} = B^{-1}A^{-1}$.

Solution *Here,*

$$AB = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 6+2 & 8+3 \\ 9+4 & 12+6 \end{bmatrix} = \begin{bmatrix} 8 & 11 \\ 13 & 18 \end{bmatrix}.$$

$$(AB)^{-1} = \begin{bmatrix} 18 & -11 \\ -13 & 8 \end{bmatrix}. \tag{2.4}$$

Again

$$A^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

and

$$B^{-1} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}.$$

Hence,

$$B^{-1}A^{-1} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 6+12 & -3-8 \\ -4-9 & 2+6 \end{bmatrix} = \begin{bmatrix} 18 & -11 \\ -13 & 8 \end{bmatrix}. \tag{2.5}$$

From (2.4)-(2.5)

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Chapter 2 Exercise

1. Define the following:
   (a). Hermitian matrix
   (b). Upper triangular matrix
   (c). Transpose of a matrix
   (d). Conjugate transpose of a matrix
   (e). Idempotent matrix
2. Find the transpose of the matrix

$$\begin{pmatrix} 2 & 1 & -1 \\ 3 & 2 & 3 \\ 1 & 4 & 2 \end{pmatrix}.$$

3. Prove that
$$(AB)^T = B^T A^T,$$
where A and B are matrices.
4. Prove that if A is a square matrix over the complex field then A can be expressed uniquely as the sum of a Hermitian matrix, and a skew-Hermitian matrix.
5. Find the inverse of following matrices
   (a). $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$
   (b). $\begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix},$
6. Given that $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$, and $B = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$, verify that $(AB)^{-1} = B^{-1}A^{-1}$.

Chapter 3 Rank of A Matrix

Introduction

- *Definitions*
- *Rank of Matrices*

3.1 Rank of Matrices

Definition 3.1

The rank *of a matrix A is the maximum number of linearly independent rows and column in the matrix.* ♣

Problem 3.1 Find the rank of the matrix $\begin{pmatrix} 6 & 2 & 0 & 4 \\ -2 & -1 & 3 & 4 \\ -1 & -1 & 6 & 10 \end{pmatrix}$.

Solution *We reduce the matrix to echelon form by the elementary row operation*

$$\begin{aligned} &\begin{pmatrix} 6 & 2 & 0 & 4 \\ -2 & -1 & 3 & 4 \\ -1 & -1 & 6 & 10 \end{pmatrix} \\ \sim &\begin{pmatrix} 6 & 2 & 0 & 4 \\ 0 & -1 & 9 & 16 \\ 0 & -4 & 36 & 64 \end{pmatrix} \qquad [r_2' = 3r_2 + r_1, r_3' = 6r_3 + r_1] \\ \sim &\begin{pmatrix} 6 & 2 & 0 & 4 \\ 0 & -1 & 9 & 16 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad [r_3' = r_3 - 4r_2,] \end{aligned} \tag{3.1}$$

This matrix is row equivalent to the given matrix and is in the row echelon form, the echelon matrix has two non-zero rows, the rank of the given matrix is 2.

Problem 3.2 Find the rank of the matrix $\begin{pmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{pmatrix}$.

Solution *We reduce the matrix to echelon form by the elementary row operation*

$$\begin{aligned}
&\begin{pmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{pmatrix} \\
\sim &\begin{pmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 0 & 4 & 4 & 1 \\ 0 & 2 & 4 & 1 \end{pmatrix} && [r_3' = r_3 - r_1, r_4' = r_4 - r_3] \\
\sim &\begin{pmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 4 & 1 \end{pmatrix} && [r_3' = 4r_2 - 3r_3, r_4' = 2r_4 - r_3] \\
\sim &\begin{pmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} && [r_4' = r_4 - r_3]
\end{aligned} \tag{3.2}$$

This matrix is row equivalent to the given matrix and is in the row echelon form, the echelon matrix has two non-zero rows, the rank of the given matrix is 2.

Chapter 3 Exercise

1. Define following
   (a). Rank.
2. Find the rank of the matrix $\begin{pmatrix} 6 & 2 & 0 & 4 \\ -2 & -1 & 3 & 4 \\ -1 & -1 & 6 & 10 \end{pmatrix}$.
3. Find the rank of the matrix $\begin{pmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{pmatrix}$.

Chapter 4 Vector Spaces

Introduction

- *Vector Space*
- *Subspace*
- *Linear Combination of Vectors*
- *Linear Dependence and Linear Independence*

4.1 Vector Space

Definition 4.1

A vector space *over an arbitrary field F is a nonempty set of vectors V, for which two operations are prescribed,*

1. Vector Addition*: If $u, v \in V$ then $u + v \in V$.*
2. Scalar Multiplication*: If $\alpha \in F$ and $u \in V$ then $\alpha u \in V$.*

The two operations are required to satisfy the axioms 4.1.

Axiom 4.1

1. *Addition is commutative: $\forall u, v \in V, \quad u + v = v + u$.*
2. *Addition is associative: $\forall u, v, w \in V, \quad (u + v) + w = u + (v + w)$.*
3. *Existence of O (zero vector): $\exists O \in V$ such that $\forall v \in V, \ \ v + O = O + v = v$.*
4. *Existence of negative: $v \in V, \exists - v \in V$, for which $v + (-v) = (-v) + v = 0$.*
5. *For each $\alpha \in F$ and $\forall u, v \in V, \quad \alpha(u + v) = \alpha u + \alpha v$.*
6. *For each $\alpha, \beta \in F$, and $\forall v \in V, \quad (\alpha + \beta)v = \alpha v + \beta v$.*
7. *For each $\alpha, \beta \in F$, and $\forall v \in V, \quad (\alpha\beta)v = \alpha\left(\beta v\right)$.*
8. *For each $v \in V$, $1v = v$, where 1 is the unite scalar and $1 \in F$*

Theorem 4.1

Let V be the set of all functions from a non-empty set S into an arbitrary field F. For any functions and any scalar $\alpha \in F$. Let $f + g \in V, \quad \forall f, g \in V$, and $\alpha f \in V, \quad \forall \alpha \in F, \quad \forall f \in V$ be defined as

$$(f + g)(x) = f(x) + g(x), \quad \forall x \in S,$$

and

$$(\alpha f)(x) = \alpha f(x), \quad \forall x \in S.$$

Prove that V is a vector space over the field F.

Proof Since S is non-empty, V is also non-empty. Now we have to show that all the axioms 4.1 of a vector space hold.

1. Let $f, g \in V$, Then
$$(f+g)(x) = f(x) + f(x) = g(x) + f(x) = (g+f)(x) \text{ for every} x \in S.$$
Thus $f + g = g + f$.
2. Let $f, g, h \in V$, then
$$((f+g)+h)(x) = (f+g)(x) + (h)(x) = (f(x)+g(x)) + h(x)$$
$$(f+(g+h))(x) = f(x) + (g+h)(x) = f(x) + (g(x)+h(x))$$
for every $x \in S$. But $f(x), g(x)$ and $h(x)$ are scalars in the field F, where addition of scalar is associative. Hence
$$(f(x)+g(x)) + h(x) = f(x) + (g(x)+h(x))$$
Accordingly, $(f+g)+h = f+(g+h)$.
3. Let O denote the zero function, $O(x) = 0$, for every $x \in S$. Then for any function $f \in V$
$$(f+O)(x) = f(x) + O(x) = f(x) + 0 = f(x)$$
for every $x \in S$. Thus $f + O = f$ and O is the zero vector in V.
4. For any function $f \in V$, let $-f$ be the function defined by $(-f)(x) = -f(x)$. Then
$$(f+(-f))(x) = f(x) + (-f)(x) = f(x) - f(x) = O = O(x)$$
for every $x \in S$. Hence $f + (-f) = O$.
5. Let $\alpha \in F$ and $f, g \in V$. Then
$$(\alpha(f+g))(x) = \alpha((f+g)(x)) = \alpha((f(x)+g(x)) = \alpha f(x) + \alpha g(x)$$
for every $x \in S$. Hence $\alpha(f+g) = \alpha f + \alpha g$.
6. Let $\alpha, \beta \in F$ and $f \in V$. Then
$$((\alpha+\beta)f)(x) = (\alpha+\beta) f(x) = \alpha f(x) + \beta f(x) = (\alpha f + \beta f)(x)$$
for every $x \in S$. Hence $(\alpha+\beta)f = \alpha f + \beta f$.
7. Let $\alpha, \beta \in F$ and $f \in V$. Then
$$((\alpha\beta)f)(x) = (\alpha\beta) f(x) = \alpha(\beta f(x)) = (\alpha(\beta f))(x)$$
for every $x \in S$. Hence $(\alpha\beta)f = \alpha(\beta f)$.
8. Let $f \in V$, then for the unite scalar $1 \in F$,
$$(1f)(x) = 1f(x) = f(x)$$
for every $x \in S$. Hence, $1f = f$.

4.2 Subspace

Definition 4.2 (Subspace)

Let W be a non-empty subset of a vector V over the field F. We call W is a subspace *of V if and only if w is a vector space over the field F under the laws of vector addition and scalar multiplication defined on V, or equivalently, W is a* subspace *of V whenever $w_1, w_2 \in W$, and $\alpha, \beta \in F$ implies that $\alpha w_1 + \beta w_2 \in W$.* ♣

Problem 4.1 Show that $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3, |x_1 - x_2 + x_3 = 0\}$ is a subspace of $\mathbb{R}^3$.

Solution *For $\bar{0} \in \mathbb{R}^3$, $\bar{0} = (0,0,0) \in S$. Since $0 - 0 + 0 = 0$. Hence, $\bar{T}$ is nonempty.*

Suppose that $\bar{u} = (x_1, x_2, x_3)$, and $\bar{v} = (x_1', x_2', x_3')$ are in T, then $x_1 - x_2 + x_3 = 0$, and $x_1' - x_2' + x_3' = 0$. Now for any scalars α, $\beta \in \mathbb{R}$, we have

$$\begin{aligned} \alpha\bar{u} + \beta\bar{v} &= \alpha(x_1, x_2, x_3) + \beta(x_1', x_2', x_3') \\ &= (\alpha x_1, \alpha x_2, \alpha x_3) + (\beta x_1', \beta x_2', \beta x_3') \\ &= (\alpha x_1 + \beta x_1', \alpha x_2 + \beta x_2', \alpha x_3 + \beta x_3') \end{aligned}$$

Also we have,

$$\begin{aligned} (\alpha x_1 + \beta x_1') - (\alpha x_2 + \beta x_2') + (\alpha x_3 + \beta x_3') &= \alpha(x_1 - x_2 + x_3) + \beta(x_1' - x_2' + x_3') \\ &= \alpha 0 + \beta 0 = 0. \end{aligned}$$

Thus $\alpha\bar{u} + \beta\bar{v} \in T$ and so T is a subspace of $\mathbb{R}^3$.

4.3 Linear Combination of vectors

Definition 4.3

Let V be a vector space over the field F and let $v_1, v_2, \ldots, v_n \in V$ then any vector $\bar{v} \in V$ is called a linear combination *of $v_1, v_2, \ldots, v_n$ if and only if there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$, such that $\bar{v} = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = \sum\limits_{k=1}^{n} \alpha_k v_k$.* ♣

Problem 4.2 Write the vector $u = (3, 9, -4, -2)$ as a linear combination of the vectors $u_1 = (1, -2, 0, 3)$, $u_2 = (2, 3, -1, 0)$, and $u_3 = (2, -1, 2, 1)$.

Solution *In order to show that u is a linear combination of u_1, u_2, and u_3, there must be a scalars α_1, α_2, and α_3 such that*

$$\begin{aligned} u &= \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 \\ (3, 9, -4, -2) &= \alpha_1(1, -2, 0, 3) + \alpha_2(2, 3, -1, 0) + \alpha_3(2, -1, 2, 1) \\ &= (\alpha_1 + 2\alpha_2 + 2\alpha_3, -2\alpha_1 + 3\alpha_2 - \alpha_3, -\alpha_2 + 2\alpha_3, 3\alpha_1 + \alpha_3) \end{aligned}$$

Equating the corresponding components and forming linear system we get,

$$\begin{array}{rcrcrcr} \alpha_1 & + & 2\alpha_2 & + & 2\alpha_3 & = & 3 \\ -2\alpha_1 & + & 3\alpha_2 & - & \alpha_3 & = & 9 \\ & - & \alpha_2 & + & 2\alpha_3 & = & -4 \\ 3\alpha_1 & & & + & \alpha_3 & = & -2 \end{array}$$

Now reduce the elementary system to echelon form by the elementary row operations, we have,

$$\begin{array}{rcrcrcr} \alpha_1 & + & 2\alpha_2 & + & 2\alpha_3 & = & 3 \\ & + & 7\alpha_2 & + & 3\alpha_3 & = & 15 \\ & - & \alpha_2 & + & 2\alpha_3 & = & -4 \\ & - & 6\alpha_2 & - & 5\alpha_3 & = & -11 \end{array} \qquad \begin{bmatrix} r_2' = r_2 + 2r_1 \\ r_4' = r_4 - 3r_1 \end{bmatrix}$$

$$\begin{array}{rcrcrcr} \alpha_1 & + & 2\alpha_2 & + & 2\alpha_3 & = & 3 \\ & + & 7\alpha_2 & + & 3\alpha_3 & = & 15 \\ & & & + & 17\alpha_3 & = & -13 \\ & & & - & 17\alpha_3 & = & 13 \end{array} \qquad \begin{bmatrix} r_3' = 7r_3 + r_2 \\ r_4' = r_4 - 6r_3 \end{bmatrix}$$

$$\begin{array}{rcrcrcr} \alpha_1 & + & 2\alpha_2 & + & 2\alpha_3 & = & 3 \\ & + & 7\alpha_2 & + & 3\alpha_3 & = & 15 \\ & & & & 17\alpha_3 & = & -13 \\ & & & & 0 & = & 0 \end{array} \qquad \begin{bmatrix} r_4' = r_4 + r_3 \end{bmatrix}$$

which provides $\alpha_3 = \frac{-13}{17}$.

$$7\alpha_2 + 3\frac{-13}{17} = 15$$
$$\alpha_2 = \frac{294}{7 \cdot 17} = \frac{42}{17}$$
$$\alpha_1 = 3 - \frac{84}{17} + \frac{26}{17} = \frac{51 - 84 + 26}{17} = \frac{-7}{17}.$$

So, we can write

$$u = \frac{-7}{17}u_1 + \frac{42}{17}u_2 + \frac{-13}{17}u_3.$$

4.4 Linear Dependence and Linear Independence

Definition 4.4

Let V be a vector space over the field F. The vectors $v_1, v_2, \ldots, v_n \in V$ are said to be linearly dependent *over F, or simply* dependent *if there exists a non-trivial linear combination of them equal to the zero vector $\bar{0}$, i.e.*

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = \bar{0}, \qquad (4.1)$$

where $\alpha_i \neq 0$ for at least one i. ♣

Definition 4.5

Let V be a vector space over the field F. The vectors $v_1, v_2, \ldots, v_n \in V$ are said to be linearly independent *over F, or simply* independent *if the only linear combination of them equal to the zero vector $\bar{0}$ is the trivial one, i.e.*

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = \bar{0}, \tag{4.2}$$

if and only if $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. ♣

Problem 4.3 Show that the set of vectors

$$\{(3,,0,1,-1),(2,-1,0,1),(1,1,1,-2)\}$$

is linearly dependent.

Solution *Form the matrix whose rows are the given vectors and reduce the matrix to row echelon form by using elementary row operations*

$$\begin{aligned}
&\begin{bmatrix} 3 & 0 & 1 & -1 \\ 2 & -1 & 0 & 1 \\ 1 & 1 & 1 & -2 \end{bmatrix} \\
\sim &\begin{bmatrix} 1 & 1 & 1 & -2 \\ 2 & -1 & 0 & 1 \\ 3 & 0 & 1 & -1 \end{bmatrix} \qquad \begin{bmatrix} r_1' = r_3 \\ r_3' = r_1 \end{bmatrix} \\
\sim &\begin{bmatrix} 1 & 1 & 1 & -2 \\ 0 & -3 & -2 & 5 \\ 0 & -3 & -2 & 5 \end{bmatrix} \qquad \begin{bmatrix} r_2' = r_2 - 2r_1 \\ r_3' = r_3 - 3r_1 \end{bmatrix} \\
\sim &\begin{bmatrix} 1 & 1 & 1 & -2 \\ 0 & -3 & -2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} r_3' = r_3 - r_2 \end{bmatrix}
\end{aligned}$$

This matrix is in row echelon form and has a zero row; hence the given vectors are linearly dependent.

Chapter 4 Exercise

1. Define the following
   (a). Vector space.
   (b). Subspace.
2. Let V be the set of all functions from a non-empty set S into an arbitrary field F. For any functions and any scalar $\alpha \in F$. Let $f + g \in V, \quad \forall f, g \in V$, and $\alpha f \in V, \quad \forall \alpha \in F, \quad \forall f \in V$ be defined as

   $$(f+g)(x) = f(x) + g(x), \quad \forall x \in S,$$

   and

   $$(\alpha f)(x) = \alpha f(x), \quad \forall x \in S.$$

   Prove that V is a vector space over the field F.

3. Write the vector $u = (3, 9, -4, -2)$ as a linear combination of the vectors $u_1 = (1, -2, 0, 3)$, $u_2 = (2, 3, -1, 0)$, and $u_3 = (2, -1, 2, 1)$.
4. Show that $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3, |x_1 - x_2 + x_3 = 0\}$ is a subspace of $\mathbb{R}^3$.
5. Determine whether the following vectors in $\mathbb{R}^3$ are linearly independent or dependent.
   (a). (1,3,2), (1,-7,-8), (2,1,-1);
   (b). (1,2,3), (1,-3,2), (2,-1,5);
   (c). (3,,0, 1, -1), (2, -1, 0, 1), (1, 1, 1, -2)

Chapter 5 Linear Transformation

Introduction

- *Introduction to Linear Transformation*
- *Properties of Linear Transformation*

5.1 Introduction to Linear Transformation

Definition 5.1

Let U and V be two vector spaces over the same vector field F. A linear transformation *of T of U into V, written as $T : U \to V$, is a transformation of U into V such that*

1. $T(u_1 + u_2) = T(u_1) + T(u_2)$ *for all* $u_1, u_2 \in U$,
2. $T(\alpha u) = \alpha T(u)$ *for all* $u \in U$ *and all* $\alpha \in F$.

♣

5.2 Properties of Linear Transformation

Theorem 5.1

If $T : U|V$ is a linear transformation then

1. $T(0) = 0$.
2. $T(-x) = -T(x), \quad \forall x \in U$.
3. $T(x - y) = T(x) - T(y), \quad \forall x, y \in U$.

♡

Proof

1. Let u be any vector in U. Since $0x = 0$, we have
$$T(0) = T(0x) = 0T(x) = 0.$$
2. $T(-x) = T((-1)x) = (-1)T(x) = -T(x)$.
3. Finally, $x - y = x + (-1)y$, thus
$$T(x - y) = T(x + (-1)y) = T(x) + T((-1)y) = T(x) + (-1)T(y) = T(x) - T(y). \tag{5.1}$$
.

Chapter 5 Exercise

1. Prove that if $T : U|V$ is a linear transformation then
   (a). $T(0) = 0$.
   (b). $T(-x) = -T(x), \quad \forall x \in U$.
   (c). $T(x - y) = T(x) - T(y), \quad \forall x, y \in U$.

Chapter 6 Eigenvalues and Eigenvectors

Introduction

- *Polynomials*
- *Eigenvalues and Eigenvectors*
- *Characteristic Polynomial and Characteristic Equation*
- *Diagonalization*
- *Cayley-Hamiltonian Theorem*

6.1 Polynomials

Definition 6.1

Let F be a field and λ be an indeterminate, then an expression of type

$$f(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n$$

where n is an integer $(n > 0)$, a_0, a_1, a_2, …, $a_n \in F$, and $a_0 \neq 0$ is known as the polynomial of degree n.

Now if A is a square matrix over F, then we define

$$f(A) = a_0A^n + a_1A^{n-1} + a_2A^{n-2} + \cdots + a_{n-1}A + a_nI$$

where I is the identity matrix.

In particular, we say that A is a root *or* zero *of the polynomial $f(\lambda)$ if $f(A) = 0$.* ♣

6.2 Eigenvalues and Eigenvectors

Definition 6.2

If A is an $n \times n$ matrix, then there is a pair (λ, v) such that

$$Av = \lambda v,$$

where λ is a scalar called an eigenvalue *of the matrix, and v is the corresponding* eigenvector *of A.* ♣

6.3 Characteristic Polynomial and Characteristic Equation

Definition 6.3

The determinant of the characteristic matrix $\lambda I - A$ is $|\lambda I - A|$ is a polynomial in λ and is called the characteristic polynomial of A. ♣

Theorem 6.1
Any square matrix A and its transpose A' have the same eigenvalues. ♡

Theorem 6.2
If A be a non-singular matrix then the eigenvalues of A^{-1} are reciprocals of the eigenvalues of A. ♡

Theorem 6.3
If the eigenvalues of A are $\lambda_1, \lambda_2, \ldots, \lambda_n$, then the eigenvalues of A^k are $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$. ♡

Theorem 6.4
The eigenvalues of a real symmetric matrix are all real. ♡

Theorem 6.5
The eigenvalues of a Hermitian matrix are all real. ♡

6.4 Diagonalization

Definition 6.4
A square matrix A is called diagonalizable *if there exists an invertible matrix P such that $P^{-1}AP$ is diagonal, the matrix P is said to diagonalize A.* ♣

6.5 Cayley-Hamiltonian Theorem

Definition 6.5
Every square matrix satisfies its own characteristic equation, i.e. if the characteristic equation of the nth order matrix A is

$$f(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n = 0,$$

then Cayley-Hamiltonian theorem states that

$$f(A) = A^n + a_1A^{n-1} + a_2A^{n-2} + \cdots + a_{n-1}A + a_nI = 0,$$

where I is the nth order unite matrix and 0 is the nth order zero matrix. ♣

Problem 6.1 Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

.

1. Find all the eigenvalues of the matrix .

2. Verify Cayley-Hamilton theorem for the matrix A.
3. Using Cayley-Hamilton theorem find the inverse of A.

Solution *The characteristic matrix of A is*

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \lambda - 1 & -2 & -3 \\ -2 & \lambda + 1 & -1 \\ -3 & -1 & \lambda - 1 \end{bmatrix}$$

The determinate of the matrix A is

$$\begin{aligned} |\lambda I - A| &= \begin{vmatrix} \lambda - 1 & -2 & -3 \\ -2 & \lambda + 1 & -1 \\ -3 & -1 & \lambda - 1 \end{vmatrix} \\ &= (\lambda - 1)\left(\lambda^2 - 1 - 1\right) + 2(-2\lambda + 2 - 3) - 3(2 + 3\lambda + 3) \\ &= (\lambda - 1)(\lambda^2 - 2) - 4\lambda - 2 - 9\lambda - 15 \\ &= \lambda^3 - 2\lambda - \lambda^2 + 2 - 10\lambda - 17 \\ &= \lambda^3 - \lambda^2 - 15\lambda - 15. \end{aligned}$$

Therefore, the characteristic equation of A is

$$\lambda^3 - \lambda^2 - 15\lambda - 15 = 0$$

by solving this we get the eigenvalues are, -2.567, -1.221, 4.788.

Now in order to verify Cayley-Hamilton theorem we have to show that

$$A^3 - A^2 - 15A - 15I = 0.$$

$$A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 3 & 8 \\ 3 & 6 & 6 \\ 8 & 6 & 11 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} 14 & 3 & 8 \\ 3 & 6 & 6 \\ 8 & 6 & 11 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 44 & 33 & 53 \\ 33 & 6 & 21 \\ 53 & 21 & 41 \end{bmatrix}$$

$$\begin{aligned} A^3 - A^2 - 15A - 15I &= \begin{bmatrix} 44 & 33 & 53 \\ 33 & 6 & 21 \\ 53 & 21 & 41 \end{bmatrix} - \begin{bmatrix} 14 & 3 & 8 \\ 3 & 6 & 6 \\ 8 & 6 & 11 \end{bmatrix} - 15 \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix} - 15 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 44 - 14 - 15 - 15 & 33 - 3 - 30 - 0 & 53 - 8 - 45 - 0 \\ 33 - 3 - 30 - 0 & 6 - 6 + 15 - 15 & 21 - 6 - 15 - 0 \\ 53 - 8 - 45 - 0 & 21 - 6 - 15 - 0 & 41 - 11 - 15 - 15 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Multiplying the above Cayley-Hamilton equation on both sides by A^{-1} we have

$$A^2 - A - 15I - 15A^{-1} = 0$$
$$\implies A^{-1} = \frac{1}{15}A^2 - \frac{1}{15}A - I$$

$$\begin{aligned}
A^{-1} &= \frac{1}{15}A^2 - \frac{1}{15}A - I \\
&= \frac{1}{15}\begin{bmatrix} 14 & 3 & 8 \\ 3 & 6 & 6 \\ 8 & 6 & 11 \end{bmatrix} - \frac{1}{15}\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \frac{1}{15}\begin{bmatrix} 14-1-15 & 3-2 & 8-3 \\ 3-2 & 6+1-15 & 6-1 \\ 8-3 & 6-1 & 11-1-15 \end{bmatrix} \\
&= \frac{1}{15}\begin{bmatrix} -2 & 1 & 5 \\ 1 & -8 & 5 \\ 5 & 5 & -5 \end{bmatrix}
\end{aligned}$$

Problem 6.2 Given that

$$A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$$

1. Find eigenvalues of the matrix A.
2. Find eigenvectors of the matrix A for corresponding eigenvalues.

Solution *The characteristic matrix of A is*

$$\lambda I - A = \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix} = \begin{bmatrix} \lambda-4 & -6 & -6 \\ -1 & \lambda-3 & -2 \\ 1 & 4 & \lambda+3 \end{bmatrix}$$

The determinate of the matrix A is

$$\begin{aligned}
|\lambda I - A| &= \begin{vmatrix} \lambda-4 & -6 & -6 \\ -1 & \lambda-3 & -2 \\ 1 & 4 & \lambda+3 \end{vmatrix} \\
&= (\lambda-4)\left(\lambda^2-9+8\right) + 6(-\lambda-3+2) - 6(-4-\lambda+3) \\
&= (\lambda-4)(\lambda^2-1) - 6\lambda - 6 + 6 + 6\lambda \\
&= (\lambda-4)(\lambda^2-1).
\end{aligned}$$

Therefore, the characteristic equation of A is

$$(\lambda-4)(\lambda^2-1) = 0$$

Which provides eigenvalues of A are $\lambda = 4$, $\lambda = -1$, and $\lambda = 1$.

Now by definition $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is an eigenvalues of A corresponding to the eigenvalue λ if and *only if X is a non-trivial solution of*

$$
\begin{aligned}
&(\lambda I - A)\, X = 0 \\
\implies &\begin{bmatrix} \lambda-4 & -6 & -6 \\ -1 & \lambda-3 & -2 \\ 1 & 4 & \lambda+3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
\tag{6.1}
$$

If $\lambda = 4$ then equation (6.1) *becomes*

$$
\begin{aligned}
&\begin{bmatrix} 0 & -6 & -6 \\ -1 & 1 & -2 \\ 1 & 4 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\sim &\begin{bmatrix} 1 & 4 & 7 \\ -1 & 1 & -2 \\ 0 & -6 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} r_1' = r_3 \\ r_3' = r_1 \end{bmatrix} \\
\sim &\begin{bmatrix} 1 & 4 & 7 \\ 0 & 1 & 1 \\ 0 & -6 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} r_2' = \frac{1}{5}(r_1 + r_2) \end{bmatrix} \\
\sim &\begin{bmatrix} 1 & 4 & 7 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} r_3' = (6r_2 + r_3) \end{bmatrix}
\end{aligned}
$$

In echelon form there are only two equations in three unknowns. Hence the system has a non-zero solution. Here x_3 is a free variable. Let $x_3 = -1$, then $x_2 = 1$, and $x_1 = 3$.

Therefore, $\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$ *is an eigenvector corresponding to the eigenvalue $\lambda = 4$.*

If $\lambda = 1$ then equation (6.1) *becomes*

$$
\begin{aligned}
&\begin{bmatrix} -3 & -6 & -6 \\ -1 & -2 & -2 \\ 1 & 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\sim &\begin{bmatrix} -3 & -6 & -6 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} r_2' = (r_1 - 3r_2) \\ r_3' = (r_2 + r_3) \end{bmatrix} \\
\sim &\begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} r_1' = r_1 + 3r_3 \\ r_3' = r_3/2 \end{bmatrix} \\
\sim &\begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} r_2' = r_3 \\ r_3' = r_2 \end{bmatrix}
\end{aligned}
$$

In echelon form there are only two equations in three unknowns. Hence the system has a non-zero solution. Here x_3 is a free variable. Let $x_3 = -1$, then $x_2 = 1$, and $x_1 = 0$.

Therefore, $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ *is an eigenvector corresponding to the eigenvalue* $\lambda = 1$.

If $\lambda = -1$ *then equation* (6.1) *becomes*

$$\begin{bmatrix} -5 & -6 & -6 \\ -1 & -4 & -2 \\ 1 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -3 & -6 & -6 \\ 0 & -14 & -4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} r_2' = 5r_2 - r_1 \\ r_3' = r_2 - r_3 \end{bmatrix}$$

In echelon form there are only two equations in three unknowns. Hence the system has a non-zero solution. Here x_3 *is a free variable. Let* $x_3 = -7$, *then* $x_2 = 2$, *and* $x_1 = 6$.

Therefore, $\begin{bmatrix} 6 \\ 2 \\ -7 \end{bmatrix}$ *is an eigenvector corresponding to the eigenvalue* $\lambda = -1$.

Chapter 6 Exercise

1. State Cayley-Hamilton theorem for matrix.
2. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

.

   (a). Find all the eigenvalues of the matrix .
   (b). Verify Cayley-Hamilton theorem for the matrix A.
   (c). Using Cayley-Hamilton theorem find the inverse of A.

3. Find the the characteristic equation of the matrix

$$A = \begin{pmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{pmatrix}$$

4. Given that

$$A = \begin{pmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{pmatrix}$$

   (a). Find eigenvalues of the matrix A.
   (b). Find eigenvectors of the matrix A for corresponding eigenvalues.

Chapter 7 Rectangular Co-ordinates

Introduction

- *Distance between two points*
- *Direction Cosines of a Line*
- *Direction Ratios of a Line*

7.1 Distance between two Points

Theorem 7.1

Distance between two points $P(x_1, y_1, z_1)$, and $Q(x_2, y_2, z_2)$ is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \tag{7.1}$$

Corollary 7.1

Distance of $P(x, y, z)$from the origin is given by

$$OP = \sqrt{x^2 + y^2 + z^2}. \tag{7.2}$$

Problem 7.1 Find the distance between the points $(5, -2, 3)$, $(-4, 3, 7)$.

Solution *The distance between the points is*

$$\sqrt{(-4-5)^2 + (3+2)^2 + (7-3)^2} = \sqrt{(-9)^2 + (5)^2 + (4)^2} = \sqrt{81 + 25 + 16} = \sqrt{122}.$$

7.2 Direction Cosine of a Line

Definition 7.1

If a given line OP makes angles α, β, and γ, with the positive direction of axes of x, y, and z respectively then $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are the direction cosines *(dcs) of the line OP, and are generally denoted by the letters, l, m, and n respectively.*

Problem 7.2 Find the direction cosines of the positive y axis.

Solution *The y axis makes with the co-ordinates axes the angles $90°$, $0°$, and $90°$, and hence dcs are* $(\cos 90°, \cos 0°, \cos 90°)$, *or* $(0, 1, 0)$.

7.2.1 Direction Cosines of a Line Joining two Points

Direction ratios of a line passing through the points $P(x_1, y_1, z_1)$, and $Q(x_2, y_2, z_2)$, let

$$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{7.3}$$

then the direction cosines are

$$\begin{aligned} l &= \frac{x_2 - x_1}{r} = \frac{a}{r}, \\ m &= \frac{y_2 - y_1}{r} = \frac{b}{r}, \\ n &= \frac{z_2 - z_1}{r} = \frac{c}{r}. \end{aligned}$$

7.3 Direction Ratios of a Line

7.3.1 Direction Ratios of a Line Joining two Points

Direction ratios of a line passing through the points $P(x_1, y_1, z_1)$, and $Q(x_2, y_2, z_2)$ are $x_2 - x_1$, $y_2 - y_1$, and $z_2 - z_1$.

Problem 7.3 Find the direction ratio and direction cosines of joining two points $(2, -3, 1)$, and $(3, -4, -5)$.

Solution *The direction ratios of a line passing through the points* $(2, -3, 1)$, *and* $(3, -4, -5)$ *are* $3 - 2 = 1$, $-4 + 3 = -1$, *and* $-5 - 1 = -6$. *We also have*

$$r = \sqrt{1^2 + (-1)^2 + (-6)^2} = \sqrt{38}.$$

Hence, the dcs are $1/\sqrt{38}$, $-1/\sqrt{38}$, *and* $-6/\sqrt{38}$.

Problem 7.4 If P, and Q are $(2, 3, -6)$, and $(3, -4, 5)$ respectively and O be the origin, find the direction cosine of OP, and OQ.

Solution *Here,* $OP = \sqrt{2^2 + 3^2 + (-6)^2} = 7$. *Hence direction cosines of* OP *are*

$$\begin{aligned} l &= \frac{2 - 0}{7} = \frac{2}{7}, \\ m &= \frac{3 - 0}{7} = \frac{3}{7}, \\ n &= \frac{-6 - 0}{7} = \frac{-6}{7}. \end{aligned}$$

Similarly, $OQ = \sqrt{3^2 + (-4)^2 + 5^2} = 5\sqrt{2}$. *Hence direction cosines of* OQ *are*

$$\begin{aligned} l &= \frac{3 - 0}{5\sqrt{2}} = \frac{3}{5\sqrt{2}}, \\ m &= \frac{-4 - 0}{5\sqrt{2}} = \frac{-4}{5\sqrt{2}}, \\ n &= \frac{5 - 0}{5\sqrt{2}} = \frac{5}{5\sqrt{2}}. \end{aligned}$$

7.3.2 Condition of Perpendicularity of two Lines

If direction cosines of two lines are given, two lines are perpendicular to each other if

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0.$$

Also if direction ratios of two lines are given, then two lines are perpendicular to each other if

$$a_1 a_2 + b_1 b_2 + c_1 c_2 = 0.$$

7.3.3 Condition of Parallelism of two Lines

If direction cosines of two lines are given, two lines are parallel to each other if

$$l_1 = l_2, m_1 = m_2, n_1 = n_2.$$

Also if direction ratios of two lines are given, then two lines are parallel to each other if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

Problem 7.5 Prove that the joining points $(2, 3, -2)$, and $(3, 1, 1)$ is parallel to the line joining joining points $(2, 1, -5)$, and $(4, -3, 1)$.

Solution *The direction ratios of a line passing through the points* $(2, 3, -2)$*, and* $(3, 1, 1)$ *are* $3-2 = 1$, $1 - 3 = -2$*, and* $1 + 2 = 3$*. Also the direction ratios of a line passing through the points* $(2, 1, -5)$*, and* $(4, -3, 1)$ *are* $4 - 2 = 2$, $-3 - 1 = -4$*, and* $1 + 5 = 6$*. Now, two lines are parallel since*

$$\frac{1}{2} = \frac{-2}{-4} = \frac{3}{6}.$$

Chapter 7 Exercise

1. Find the distance between the points $(5, -2, 3)$, $(-4, 3, 7)$.
2. Find the direction cosines of the positive y axis.
3. Find the direction ratio and direction cosines of joining two points $(2, -3, 1)$, and $(3, -4, -5)$.
4. If P, and Q are $(2, 3, -6)$, and $(3, -4, 5)$ respectively and O be the origin, find the direction cosine of OP, and OQ.
5. Prove that the joining points $(2, 3, -2)$, and $(3, 1, 1)$ is parallel to the line joining joining points $(2, 1, -5)$, and $(4, -3, 1)$.

Chapter 8 Equations of Planes

Introduction

8.1 Introduction to Plane

Definition 8.1

A plane is a surface such that if any points are taken on it, the straight line joining them lies wholly and the surface i.e. every points on the line joining the two points will be on the plane. ♣

8.2 General Equation of Plane

Theorem 8.1

The general equation of a plane is given by $ax + by + cz + d = 0$. ♡

8.2.1 General Equation of a Plane Passes Through a Point

Corollary 8.1

The general equation of a plane that passes through a given point (x_1, y_1, z_1) *is given by* $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$. ♡

Corollary 8.2

The general equation of a plane that passes through the origin is given by $ax + by + cz = 0$. ♡

8.2.2 General Equation of a Plane Passes Through three Points

Theorem 8.2

The plane that passes through three given points (x_1, y_1, z_1), (x_2, y_2, z_2), *and* (x_3, y_3, z_3) *is given* *by*

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

Problem 8.1 Find the equation of the plane through the points, $(2, 1, -3), (3, -1, 4), (7, 5, 6)$.

Solution *The plane passes through the point* $(2, 1, -3)$ *is given by*

$$a(x-2) + b(y-1) + c(z+3) = 0 \tag{8.1}$$

Since (8.1) *passes through (3,-1,4) and (7,5,6), we have*

$$a(3-2) + b(-1-1) + c(4+3) = 0 \implies a - 2b + 7c = 0, \tag{8.2}$$

$$a(7-2) + b(5-1) + c(6+3) = 0 \implies 5a + 4b + 9c = 0. \tag{8.3}$$

Solving (8.2)-(8.3) *for* a*,* b*, and* c *by cross multiplication, we have*

$$\begin{array}{rcccc} & \frac{a}{-18-28} & = & \frac{b}{35-9} & = & \frac{c}{4+10} \\ \implies & \frac{a}{-46} & = & \frac{b}{26} & = & \frac{c}{14} \\ \implies & \frac{a}{-23} & = & \frac{b}{13} & = & \frac{c}{7} \end{array}$$

Putting these value in (8.1)*, we get*

$$-23(x-2) + 13(y-1) + 7(z+3) = 0$$

$$-23x + 13y + 7z = -46 + 13 - 21 = -54.$$

Problem 8.2 Show that four points (0,-1,-1), (4,5,1), (3,9,4), and (-4,4,4) lie on a plane.

Solution *Four points are co-planer, since*

$$
\begin{aligned}
&\begin{vmatrix} 0 & -1 & -1 & 1 \\ 4 & 5 & 1 & 1 \\ 3 & 9 & 4 & 1 \\ -4 & 4 & 4 & 1 \end{vmatrix} \\
=& \begin{vmatrix} 0 & 0 & 0 & 1 \\ 4 & 6 & 2 & 1 \\ 3 & 10 & 5 & 1 \\ -4 & 5 & 5 & 1 \end{vmatrix} \qquad \begin{bmatrix} c_2' = c_2 + c_4 \\ c_3' = c_3 + c_4 \end{bmatrix} \\
=& -\begin{vmatrix} 4 & 6 & 2 \\ 3 & 10 & 5 \\ -4 & 5 & 5 \end{vmatrix} \\
=& -\begin{vmatrix} 2 & 3 & 1 \\ 3 & 10 & 5 \\ -4 & 5 & 5 \end{vmatrix} \qquad \begin{bmatrix} r_1' = r_1/2 \end{bmatrix} \\
=& -\begin{vmatrix} 2 & 3 & 1 \\ 7 & 5 & 0 \\ 14 & 10 & 0 \end{vmatrix} \qquad \begin{bmatrix} r_3' = -r_3 + 5r_1 \\ r'4 = -r_4 + 5r_1 \end{bmatrix} \\
=& -(70 - 70) = 0.
\end{aligned}
$$

Hence, four points are co-planer.

8.3 Equation of a Plane Passes Through Planes

Theorem 8.3

Any plane passes through the intersection of two planes $a_1x + b_1y + c_1z + d_1 = 0$, *and* $a_2x + b_2y + c_2z + d_2 = 0$ *is* $a_1x + b_1y + c_1z + d_1 + k\,(a_2x + b_2y + c_2z + d_2) = 0$. ♡

Values of k can be found through any other condition.

Problem 8.3 Find the equation of the plane passing through the intersection of two planes $x + 2y + 3z + 4 = 0$, and $4x + 3y + 2z + 1 = 0$, and the point $(1, 2, 3)$.

Solution *Any plane passing through the intersection of two planes* $x + 2y + 3z + 4 = 0$, *and* $4x + 3y + 2z + 1 = 0$, *is*

$$x + 2y + 3z + 4 + k\,(4x + 3y + 2z + 1) = 0. \tag{8.4}$$

Since, the plane (8.4) *passes through* $(1, 2, 3)$ *then we have,*

$$
\begin{aligned}
& 1 + 2 \cdot 2 + 3 \cdot 3 + 4 + k\,(4 + 3 \cdot 2 + 2 \cdot 3 + 1) = 0 \\
\implies \quad & k = \frac{-18}{17}
\end{aligned}
$$

Now, putting value of k in (8.4)*, we get the desire equation*

$$\begin{aligned} & x+2y+3z+4-\frac{18}{17}(4x+3y+2z+1)=0 \\ \Longrightarrow\quad & 17x+34y+51z+68-72x-54y-36z-18=0 \\ \Longrightarrow\quad & 55x+20y-15z-50=0 \\ \Longrightarrow\quad & 11x+4y-3z=10 \end{aligned}$$

Problem 8.4 Find the equation of the plane through the points $(2,2,1)$, and $(9,3,6)$, and perpendicular to the plane $2x+6y+6z=9$.

Solution *Any plane passes through the point* $(2,2,1)$ *is*

$$a(x-2)+b(y-2)+c(z-1)=0. \tag{8.5}$$

Since it passes through $(9,3,6)$*, then we have,*

$$\begin{aligned} & a(9-2)+b(3-2)+c(6-1)=0 \\ \Longrightarrow\quad & 7a+b+5c=0. \end{aligned} \tag{8.6}$$

The plane (8.5) *is perpendicular to* $2x+6y+6z=9$*, hence*

$$2a+6b+6c=0. \tag{8.7}$$

From (8.6)-(8.7) *by cross multiplication.*

$$\begin{aligned} & \tfrac{a}{6-30} = \tfrac{b}{10-42} = \tfrac{c}{42-2} \\ \Longrightarrow\quad & \tfrac{a}{-24} = \tfrac{b}{-32} = \tfrac{c}{40} \\ \Longrightarrow\quad & \tfrac{a}{3} = \tfrac{b}{4} = \tfrac{c}{-5} \end{aligned}$$

Putting the values of a, b, c in (8.5)*, we get the required equation*

$$\begin{aligned} & 3(x-2)+4(y-2)-5(z-1)=0 \\ \Longrightarrow\quad & 3x+4y-5z=9. \end{aligned}$$

8.4 Equation of a Plane Passes Through Lines

Problem 8.5 Find the equation of the plane passing through the line $\frac{x-2}{3}=\frac{y-3}{5}=\frac{z}{7}$, and the point $(1,-2,3)$.

Solution *Any plane passes through the given line is*

$$5(x-2)-3(y-3)=k\left(7(y-3)-5z\right). \tag{8.8}$$

Now this plane also passes through the point $(1,-2,3)$ *then we have,*

$$k=\frac{5(1-2)-3(-2-3)}{7(-2-3)-5(3)}=\frac{10}{-50}=\frac{-1}{5}.$$

Putting this value of k in (8.8)

$$\begin{aligned} & 5\left(5(x-2)-3(y-3)\right)+7(y-3)-5z=0 \\ \Longrightarrow\quad & 25x-8y-5z-26=0. \end{aligned}$$

Problem 8.6 Find the equation of the plane passing through the point $(1,-2,1)$ and perpendicular to the line with direction ratios 2, 3, 5.

Solution *Any plane passes through the point* $(1, -2, 1)$ *is*

$$a(x-1)+b(y+2)+c(z-1)=0. \tag{8.9}$$

This plane is also perpendicular to the line with direction ratios 2, 3, 5. Hence,

$$\begin{array}{rcrcrcl} 3a & - & 2b & & & = & 0 \\ & & 5b & - & 3c & = & 0 \end{array}$$

Solving for $a, b,$ *and* c, *by cross multiplication, provide us*

$$\frac{a}{6-0}=\frac{b}{0+9}=\frac{c}{15}$$
$$\frac{a}{2}=\frac{b}{3}=\frac{c}{5}$$

Putting the values of $a, b,$ *and* c *in* (8.9)*, we get the required equation*

$$2(x-1)+3(y+2)+5(z-1)=0$$
$$2x+3y+5z=1. \tag{8.10}$$

8.5 Distance from a Plane

Problem 8.7 Find the distance of the points $(2, -1, 5)$ from the plane $3x-2y+6z+8=0$.

Solution *The equation of the plane is* $3x-2y+6z+8=0$. *Its distance from* $(2, -1, 5)$ *is*

$$\frac{3(2)-2(-1)+6(5)+8}{\sqrt{3^2+2^2+6^2}}=\frac{46}{7}.$$

Problem 8.8 Find the distance of the point $(2, 0, 1)$, and $(3, -3, 2)$ from the plane $x-2y+z=6$, and find whether the two points lie on the same side or opposite sides of the plane.

Solution *The equation of the plane is* $x-2y+z-6=0$. *Its distance from* $(2, 0, 1)$ *is*

$$\frac{2-2(0)+1-6}{\sqrt{1^2+2^2+1^2}}=\frac{-3}{\sqrt{6}}.$$

Its distance from $(3, -3, 2)$ *is*

$$\frac{3-2(-3)+2-6}{\sqrt{1^2+2^2+1^2}}=\frac{5}{\sqrt{6}}.$$

The two results are of opposite signs. Therefore the two points lie on opposite side of the plane.

Chapter 8 Exercise

1. Find the equation of the plane through the points, $(2, 1, -3), (3, -1, 4), (7, 5, 6)$.
2. Show that the four points (0,-1,-1), (4,5,1), (3,9,4), and (-4,4,4) lie on a plane.
3. Find the equation of the plane passing through the intersection of two planes $x+2y+3z+4=0$, and $4x+3y+2z+1=0$, and the point $(1, 2, 3)$.
4. Find the equation of the plane passing through the line $\frac{x-2}{3}=\frac{y-3}{5}=\frac{z}{7}$, and the point $(1, -2, 3)$.
5. Find the equation of the plane passing through the point $(1, -2, 1)$ and perpendicular to the line with direction ratios 2, 3, 5.

6. Find the distance of the points $(2,-1,5)$ from the plane $3x-2y+6z+8=0$.
7. Find the distance of the point $(2,0,1)$, and $(3,-3,2)$ from the plane $x-2y+z=6$, and find whether the two points lie on the same side or opposite sides of the plane.

Chapter 9 Equations of Straight Lines

Introduction

9.1 General Equations of Straight Lines

$$a_1x + b_1y + c_1z + d_1 = 0 = a_2x + b_2y + c_2z + d_2 \tag{9.1}$$

9.2 Symmetric form Equations of Straight Lines

The equation of a straight line passing through point (x_1, y_1, z_1) and in a given direction is given by

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r \tag{9.2}$$

If direction ration is given then (9.2) can be written as

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \tag{9.3}$$

The equation of a straight line passing through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} = \frac{z - z_1}{z_1 - z_2}. \tag{9.4}$$

9.3 Two Lines are Perpendicular

Two lines are perpendicular if

$$l_1l_2 + m_1m_2 + n_1n_2 = 0, \tag{9.5}$$

or

$$a_1a_2 + b_1b_2 + c_1c_2 = 0. \tag{9.6}$$

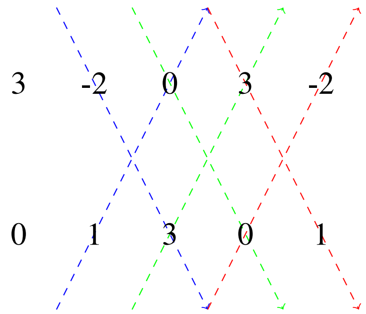
Problem 9.1 Show that the lines $3x - 2y + 13 = 0, \quad y + 3z - 26 = 0$, and $\frac{x+4}{5} = \frac{y-1}{-3} = \frac{z-3}{1}$ are perpendicular.

Solution *The direction ratios of the first line are*

(-6,0), (0-9), (3-0); or, -6, -9, 3; or -2, -3, 1.

The direction ratios of second line are 5, -3, 1.

Now for perpendicular $a_1a_2 + b_1b_2 + c_1c_2$ *is equal to zero, but* $(-2)(5) + (-3)(-3) + (1)(1) = 0.$



Hence, the lines are perpendicular.

Problem 9.2 Find the equations of line perpendicular to both the line $\frac{x-1}{1} = \frac{y-1}{2} = \frac{z+2}{3}$, $\frac{x+2}{2} = \frac{y-5}{-1} = \frac{z+3}{2}$ and passing through their intersection.

Solution *Let*

$$\frac{x-1}{1} = \frac{y-1}{2} = \frac{z+2}{3} = r_1$$
$$\implies x = r_1 + 1, y = 2r_1 + 1, z = 3r_1 - 2. \tag{9.7}$$

Also let

$$\frac{x+2}{2} = \frac{y-5}{-1} = \frac{z+3}{2} = r_2$$
$$\implies x = 2r_2 - 2, y = -r_2 + 5, z = 2r_2 - 3. \tag{9.8}$$

If the lines meet then

$$r_1 + 1 = 2r_2 - 2, 2r_1 + 2 = -r_2 + 5, 3r_1 - 2 = 2r_2 - 3$$
$$\implies r_1 = 1, r_2 = 2.$$

Hence, the point of intersection of the lines is (2,3,1).

Let l, m, n be the direction ratios of the given lines then

$$l + 2m + 3n = 0 \tag{9.9}$$
$$2l - m + 2n = 0. \tag{9.10}$$

Solving,

$$\frac{l}{7} = \frac{m}{4} = \frac{n}{-5}.$$

Hence, equation of the line is

$$\frac{x-2}{7} = \frac{y-3}{4} = \frac{z-1}{-5}.$$

9.4 Two Lines are Parallel

Two lines are parallel if

$$\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}, \tag{9.11}$$

or

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}. \tag{9.12}$$

9.5 Shortest Distance between two Lines

Problem 9.3 Find the shortest distance between the lines

$$\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}, \qquad \frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}.$$

Find also the equation and the points in which it meets the given lines.

Solution *Let*

$$\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}=r_1$$
$$\implies x=3r_1+3, y=-r_1+8, z=r_1+3. \tag{9.13}$$

Also let

$$\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}=r_2$$
$$\implies x=-3r_2-3, y=2r_2-7, z=4r_2+6. \tag{9.14}$$

Let $L(3r_1+3,-r_1+8,r_1+3)$ *and* $M(-3r_2-3,2r_2-7,4r_2+6)$ *be any two points of the given line then the directions ratios of the line* LM *are*

$$3r_1+3r_2+6, -r_1-2r_2+15, r_1-4r_2-3 \tag{9.15}$$

If the line is perpendicular to both the lines, then we have

$$3(3r_1+3r_2+6)-1(-r_1-2r_2+15)+1(r_1-4r_2-3)=0$$
$$\implies 11r_1+7r_2=0 \tag{9.16}$$

Similarly,

$$-3(3r_1+3r_2+6)+2(-r_1-2r_2+15)+4(r_1-4r_2-3)=0$$
$$\implies 7r_1+29r_2=0 \tag{9.17}$$

Solving (9.16)-(9.17)*, we get*

$$r_1=r_2=0$$

Hence the points L*, and* M *are respectively* $(3,8,3)$*, and* $(-3,-7,6)$*. The equation of the line* LM*, which is SD is given by*

$$\frac{x-3}{3+3}=\frac{y-8}{8+7}=\frac{z-3}{3-6}$$
$$\implies \frac{x-3}{6}=\frac{y-8}{15}=\frac{z-3}{-3}$$
$$\implies \frac{x-3}{2}=\frac{y-8}{5}=\frac{z-3}{-1}$$

The length of S.D. between L *and* M

$$=\sqrt{(3+3)^2+(8+7)^2+(3-6)^2}=\sqrt{6^2+15^2+3^2}=3\sqrt{30}.$$

Problem 9.4 Find the equation to the planes through the points $(1,0,-1)$ and the lines $4x-y-13=0=3y-4z-1$ and $y-2z+2=0=x-5$ and show that the equations to the line through the given point which intersects the two given lines can be written as $x=y+1=z+2$.

Solution *Any plane through* $4x-y-13=0=3y-4z-1$ *is given by*

$$4x-y-13+k_1(3y-4z-1)=0.$$

These plane passes through the points $(1, 0, -1)$,

$$k_1 = \frac{-4+13}{4-1} = 3.$$

Hence, the given equation is

$$\begin{aligned} 4x - y - 13 + 3(3y - 4z - 1) &= 0 \\ \implies 4x + 8y - 12z - 16 &= 0 \\ \implies x + 2y - 3z - 4 &= 0. \end{aligned} \tag{9.18}$$

Again any plane through $y - 2z + 2 = 0 = x - 5$ *is given by*

$$y - 2z + 2 + k_2(x-5) = 0.$$

These plane passes through the points $(1, 0, -1)$,

$$k_2 = \frac{-2-2}{1-5} = 1.$$

Hence, the given equation is

$$\begin{aligned} y - 2z + 2 + (x-5) &= 0 \\ \implies x + y - 2z - 3 &= 0. \end{aligned} \tag{9.19}$$

Subtracting (9.19) *from* (9.18), *we have*

$$\begin{aligned} y - z - 1 &= 0 \\ \implies y + 1 &= z + 2 \end{aligned} \tag{9.20}$$

Putting the value of z *in* (9.19)

$$\begin{aligned} x + 2y - 3(y-1) - 4 &= 0 \\ \implies x - y - 1 &= 0 \\ \implies x &= y + 1 \end{aligned} \tag{9.21}$$

From (9.20)-(9.21), *we have*

$$x = y + 1 = z + 2. \tag{9.22}$$

Chapter 9 Exercise

1. Write the condition for which two straight lines are perpendicular to each other.
2. Show that the lines $3x - 2y + 13 = 0, \quad y + 3z - 26 = 0$, and $\frac{x+4}{5} = \frac{y-1}{-3} = \frac{z-3}{1}$ are perpendicular.
3. Find the equations of line perpendicular to both the line $\frac{x-1}{1} = \frac{y-1}{2} = \frac{z+2}{3}$, $\frac{x+2}{2} = \frac{y-5}{-1} = \frac{z+3}{2}$ and passing through their intersection.
4. Find the shortest distance between the lines
$$\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}, \qquad \frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}.$$
Find also the equation and the points in which it meets the given lines.
5. Find the shortest distance between the lines $x + a = 2y = -12z$, and $x = y + 2a = 6z - 6a$.
6. Find the equation to the planes through the points $(1, 0, -1)$ and the lines $4x - y - 13 = 0 = 3y - 4z - 1$ and $y - 2z + 2 = 0 = x - 5$ and show that the equations to the line through the given point which intersects the two given lines can be written as $x = y + 1 = z + 2$.

Chapter 10 Harmonic Functions

Introduction

- *Laplacian Equation in 2D*
- *Laplacian Equation in 3D*
- *Harmonic Function*

10.1 Laplacian Equation in 2D

Definition 10.1

An equation having the second-order partial derivatives of the form

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{10.1}$$

is called the Laplace equation; where ∇^2 is called the Laplacian operator, and $\nabla^2 u$ is called the Laplacian of u. ♣

10.1.1 Polar form of Laplacian Equation

In polar form Laplacian equation can be written as,

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0. \tag{10.2}$$

10.2 Laplacian Equation in 3D

Also similarly Laplacian equation in 3D can be defined

Definition 10.2

An equation having the second-order partial derivatives of the form

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0. \tag{10.3}$$

is called the Laplace equation in 3D; where ∇^2 is called the Laplacian operator, and $\nabla^2 u$ is called the Laplacian of u. ♣

10.2.1 Cylindrical form of Laplacian Equation

In cylindrical coordinates Laplacian equation can be written as

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0. \tag{10.4}$$

10.2.2 Spherical form of Laplacian Equation

In spherical coordinates Laplacian equation can be written as

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2 \sin^2\phi}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\phi}{r^2}\frac{\partial u}{\partial \phi} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} = 0. \tag{10.5}$$

10.3 Harmonic Function

Definition 10.3

A function $u(x, y)$ is known as harmonic function when it is twice continuously differentiable and also satisfies the Laplace equation i.e.

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{10.6}$$

♣

10.3.1 Properties of Harmonic Function

Properties of Harmonic function are given below

1. If $f(z) = u(x, y) + iv(x, y)$ is analytic on a region A then both u and v are harmonic functions on A.
2. If $u(x, y)$ is harmonic on a connected region A, then u is the real part of an analytic function $f(z) = u(x, y) + iv(x, y)$.
3. If u and v are the real and imaginary parts of an analytic function, then we say u and v are harmonic conjugates.
4. The sum of two harmonic functions is a harmonic function.
5. An arbitrary pair of harmonic functions u and v need not be conjugated unless u + iv is an analytic function.

Chapter 10 Exercise

1. Define following
   (a). Harmonic function
2. Write the properties of harmonic function.
3. Write the Laplacian equation in Cartesian form.
4. Write the Laplacian equation in polar form.
5. Write the Laplacian equation in cylindrical form.
6. Write the Laplacian equation in spherical form.