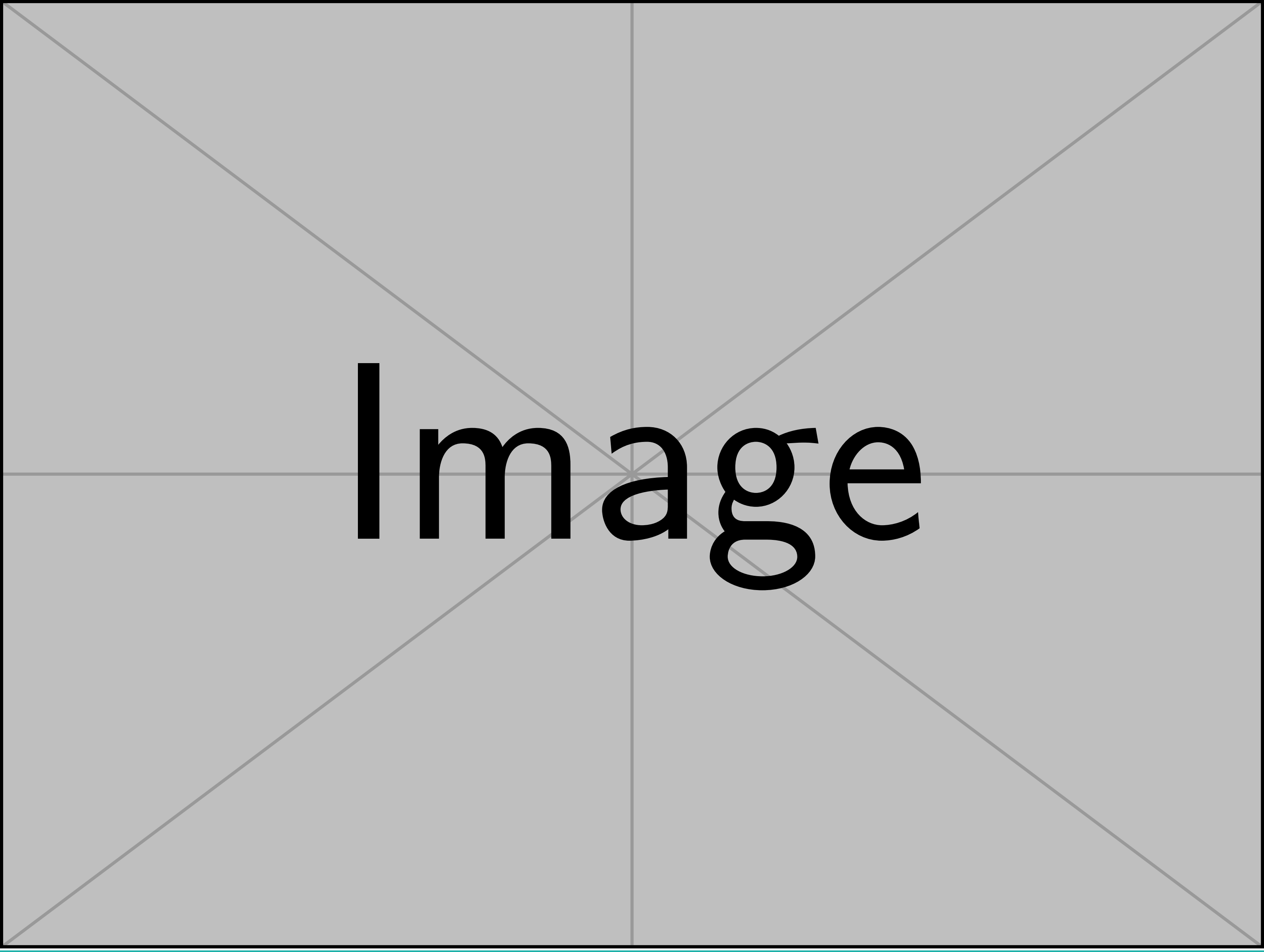

A Note on Differential & Integral Calculus

Mathematics I

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Victory won't come to us unless we go to it.

Contents

Syllabus

0.1 Syllabus

0.1.1 Differential Equation

Limit, continuity and differentiability,

successive differentiation of various types of functions, Leibnit'z theorem, Rolle's theorem, Mean Value theorem, expansion in finite and infinite forms, Lagrange's form of remainder, Cauchy's form of remainder (expansion o remainder), expansions of functions differentiation and integration, indeterminate form, Cartesian differentiation, Euler's theorem, tangent and normal, sub tangent and subnormal in cartesian and polar coordinates, maxima and minima of functions of single variables, curvature, asymptotes.

0.1.2 Integral Calculus

Definition of integrations, integration by the method of substitution, integration by parts, standard integrals,

integration by the method of successive reduction, definite integrals and its use in summing series,

Walli's formula, improper integrals, beta function and gamma function, multiple integral and its application, area, volume of solid revolution, area under a plain curve in Cartesian and polar coordinates, area of the region enclosed by two curves in Cartesian and polar coordinates, arc lengths of curves in Cartesian and polar coordinates.

0.2 Text & References Books

0.2.1 From National University

1. A text Book of Differential Calculus – Rahman and Bhattachrjee.
2. Differential Calculus – Shanti Narayan.
3. Differential Calculus – Dr. B. D. Sharma.
4. Differential Calculus – Das and Mukhajee

0.2.2 From Instructor

1. Calculus with Analytic Geometry- George Simmons - Second Edition.
2. A text Book of Differential Calculus – Rahman and Bhattachrjee.
3. A text Book of Integral Calculus – Rahman and Bhattachrjee.

0.2.3 Quick Review

1. Engineering Mathematics - John Bird- Sixth Edition.

0.3 Marks Distribution

In-course will be conducted by course teacher.

Attendance	In-course	Final Exam	Total
10	20	70	100

0.3.1 Marks Distribution for Final Exam

Time: 3:00 Hours

Full Marks: 70

Section	Question Type	No. of Question	Have to Answer	Mark per Question	Marks
Section-A	Brief questions	8	5	2	10
Section-B	Short questions	8	5	4	20
Section-C	Broad questions	7	4	10	40
				Total	70

Chapter 1 Preliminary Mathematics

Introduction

- *Number Systems*
- *Simultaneous Equations*
- *Quadratic Equations*
- *Inequalities*
- *Sequences & Series*
- *Permutations & Combinations*
- *The Binomial Theorem*
- *Trigonometry*

1.1 Numbers Systems

1.1.1 Real numbers

Definition 1.1

The numbers $1, 2, 3, \dots$ *are known as the* natural, *or* counting *numbers and the set of integers is denoted by* $\mathbb{N}$, *i.e.*

$$\mathbb{N} = \{1, 2, 3, \dots\}.$$

♣

Natural numbers are generally used for counting ("scores in sports"), ordering ("ranking in sports"), and labeling ("sports jersey numbers"). Numbers used for counting, ordering and labeling are called cardinal, ordinal, and nominal numbers respectively.

Definition 1.2

The natural numbers with zero are often called whole numbers *is denoted by* N_0,

$$\mathbb{N}_0 = \{0, 1, 2, 3, \dots\} = \{0\} \cup \mathbb{N}.$$

♣

Whole numbers are used where beginning from zero is important like in some programming languages and also used in time.

Definition 1.3

The natural numbers, their negatives and zero form the set of integers *and denoted by* $\mathbb{Z}$,

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}.$$

♣

The negative numbers are the additive inverses of the corresponding positive (natural) numbers.

Definition 1.4

Numbers which are quotients,vv or can be expressed in the form $\frac{p}{q}$, *where* $p, q \in \mathbb{Z}$ *with* $q \neq 0$, *is called* rational *numbers, and is denoted by* $\mathbb{Q}$,

$$\mathbb{Q} = \{r | r = \frac{p}{q}, \text{ where } p, q \in \mathbb{Z} \ \& \ q \neq 0\}.$$

♣

Definition 1.5

A number which represents a certain length on a straight line but can not be expressed in the form $\frac{p}{q}$, where $p, q \in \mathbb{Z}$ with $q \neq 0$, is called irrational *numbers, and is denoted by $\bar{\mathbb{Q}}$.* ♣

Some important irrational numbers are π, e, and square roots of some integers.

Definition 1.6

Rational and irrational numbers together form the continuum of real *numbers or set of real numbers is denoted by $\mathbb{R}$,*

$$\mathbb{R} = \mathbb{Q} \cup \bar{\mathbb{Q}}.$$

♣

It can be easily observed that

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

The rational numbers $\mathbb{Q}$ are countable but infinite also called countably infinite, where irrational numbers are not countable and also infinite.

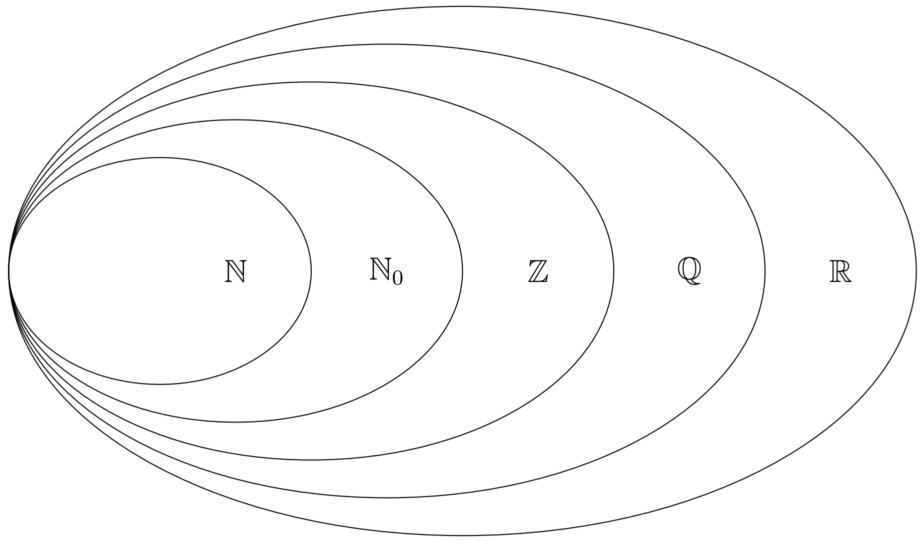


Figure 1.1: The rational numbers $\mathbb{Q}$ are included in the real numbers $\mathbb{R}$, while themselves including the integers $\mathbb{Z}$, which in turn include the natural numbers $\mathbb{N}$.

Above all the sets are infinite and $\mathbb{R}$ is the largest one.

1.2 Simultaneous Equations

Equations that have to be solved together to find the unique values of the unknown quantities, which are true for each of the equations, are called simultaneous equations.

There are several methods of solving simultaneous equations analytically are:

1. by substitution,
2. by elimination,
3. by graph,
4. by determinants, and

5. by matrices.

1.2.1 Solving by Determinants

Determinants is the best method for solving simultaneous equation of two variables, and the worst method for solving more than two variables (as graph is not suitable for more than two variables). Practically, matrices are used to solve linear systems (simultaneous equations) more than two variables, you may learn it later. Here, we will discuss only the determinants method.

A general simultaneous equations can be written as

$$a_1x + b_1y = c_1 \tag{1.1}$$

$$a_2x + b_2y = c_2 \tag{1.2}$$

then the solution of the equations (1.1)-(1.2) are given by

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1} \tag{1.3}$$

$$y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{c_2a_1 - c_1a_2}{a_1b_2 - a_2b_1} \tag{1.4}$$

If the given equations are given as follows

$$\frac{a_1}{x} + \frac{b_1}{y} = c_1 \tag{1.5}$$

$$\frac{a_2}{x} + \frac{b_2}{y} = c_2 \tag{1.6}$$

then the solution of the equations (1.5)-(1.6) can be written as

$$\frac{1}{x} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1} \tag{1.7}$$

$$\frac{1}{y} = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{c_2a_1 - c_1a_2}{a_1b_2 - a_2b_1} \tag{1.8}$$

If the given equations are given as follows

$$\frac{p}{a_1x + b_1y} = c_1 \tag{1.9}$$

$$\frac{q}{a_2x + b_2y} = c_2 \tag{1.10}$$

then equations (1.9)-(1.10) can be rewritten in the form (1.1)-(1.2) and then can be solved by (1.3)-(1.4)

1.3 Quadratic Equations

An equation is a statement that two quantities are equal and to 'solve an equation' means 'to find the value of the unknown'. The value of the unknown is called the root of the equation.

A quadratic equation is one in which the highest power of the unknown quantity is 2. A general quadratic equation can be written as

$$ax^2 + bx + c = 0. \tag{1.11}$$

There are four methods of solving quadratic equations. These are:

1. by factorization (where possible)
2. by 'completing the square'
3. by using the 'quadratic formula', and
4. by graphically.

1.3.1 Solving by Quadratic Formula

If $ax^2 + bx + c = 0$ then the solution is given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{\Delta}}{2a}. \tag{1.12}$$

Here, $\Delta = b^2 - 4ac$. $\Delta = b^2 - 4ac = 0$ is also called the characteristic equation, which determine the characteristics of the equation.

$$\text{solutions are} \begin{cases} \text{conjugate complex} & \Delta < 0 \\ \text{real and equal} & \Delta = 0 \\ \text{real and unequal} & \Delta > 0. \end{cases} \tag{1.13}$$

Along with this if $b = 0$ then the solution set will be purely imaginary $(\Delta < 0)$ or real $(\Delta > 0)$ and also one solution will be negative to other solution i.e. if x_1 and x_2 are two solution then $x_1 = -x_2$.

1.4 Inequalities

An inequality is any expression involving one of the symbols $<$, $>$, $\leq$ or $\geq$.

- $p < q$ means p is less than q
- $p > q$ means p is greater than q
- $p \leq q$ means p is less than or equal to q
- $p \geq$ means p is greater than or equal to q.

To solve an inequality means finding all the values of the variable for which the inequality is true. Two simple rules for solving inequality,

1. If $k \in \mathbb{R}$ is added to both sides of an inequality, the inequality still remains, i.e.

$$ax + b > c \implies ax + b + k > c + k$$
$$ax + b < c \implies ax + b + k < c + k$$
$$ax + b \geq c \implies ax + b + k \geq c + k$$
$$ax + b \leq c \implies ax + b + k \leq c + k$$

2. If $k > 0$ is a real number multiplied both sides of an inequality, the inequality still remains the same.

$$ax + b > c \implies k(ax + b) > ck$$
$$ax + b < c \implies k(ax + b) < ck$$
$$ax + b \geq c \implies k(ax + b) \geq ck$$
$$ax + b \leq c \implies k(ax + b) \leq ck$$

3. If $k < 0$ is a real number multiplied both sides of an inequality, the inequality is reversed.

$$ax + b > c \implies k(ax + b) < ck$$
$$ax + b < c \implies k(ax + b) > ck$$
$$ax + b \geq c \implies k(ax + b) \leq ck$$
$$ax + b \leq c \implies k(ax + b) \geq ck$$

1.4.1 Modulus

The modulus of a number is the size of the number, regardless of sign. Vertical lines enclosing the number denote a modulus. Mathematically,

$$|x| = \begin{cases} -x & x < 0 \\ 0 & x = 0 \\ x & x > 0. \end{cases} \tag{1.14}$$

Example 1.1 By using (1.14) one can show that

$$|t| \leq 5 \implies -5 \leq t \leq 5.$$

Example 1.2 For an inequality, we can write,

$$\begin{aligned} |ax + b| &\leq c \\ \implies -c \leq ax + b &\leq c \\ \implies -b - c \leq ax &\leq -b + c \\ \implies \frac{-b - c}{a} \leq x &\leq \frac{-b + c}{a} \qquad [\text{if } a > 0]. \end{aligned} \tag{1.15}$$

1.5 Sequences & Series

A sequence is a order set where each element can be indexed by natural numbers. Deepening on the context it may begin from zero or one.

Definition 1.7

A sequence *is a function of natural numbers, denoted by* $\{a_n\}$. ♣

There are two sequence Arithmetic and Geometric progression, are most used. There are also other important sequence, some of the sequences are available in nature.

Example 1.3 One of the most important sequence is Fibonacci sequence, which is as follows,

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots \tag{1.16}$$

The sequence is defined as

$$F_0 = 0, \qquad F_1 = 1, \tag{1.17}$$

$$F_n = F_{n-1} + F_{n-2}, \qquad n > 1. \tag{1.18}$$

Let a_n be a given sequence of real or complex numbers, and form a new sequence s_n as follows:

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k \qquad [n = 1, 2, 3, \dots] \tag{1.19}$$

Definition 1.8

The ordered pair of sequences of $(\{a_n\}, \{s_n\})$ *is called an* infinite series. *The number* s_n*, is called the* n*th partial sum of the series. The series is said to converge or to diverge according as* $\{s_n\}$ *is convergent or divergent.* ♣

1.5.1 Arithmetic Progression

Definition 1.9

Arithmetic progression *is a sequence where common difference* (d) *between two successive terms is constant.* ♣

Example 1.4

$$a + (a + d) + (a + 2d) + (a + 3d) + \cdots + (a + (n - 1)d) \tag{1.20}$$

(1.20) is a AP, where d is the common difference. kth element is $a_k = a + (k - 1)d$. The sum of the series s_n is given by

$$s_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}[2a + (n - 1)d] \tag{1.21}$$

1.5.2 Geometric Progression

Definition 1.10

Geometric progression *is a sequence where common ratio* (r) *between two successive term is constant.* ♣

Example 1.5

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} \tag{1.22}$$

(1.22) is a GP, where r is the common ratio. kth element is $a_k = ar^{k-1}$. The sum of the series s_n is given by

$$s_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r} \qquad [r \neq 1] \tag{1.23}$$

if $r < 1$ then $r^n \to 0$ as $n \to \infty$, which (1.23) provide us

$$s_\infty = \frac{a}{1-r}, \tag{1.24}$$

this is called sum of infinite series.

1.6 Permutations & Combinations

1.6.1 Combinations

Let we are interested to make all teams of 3 digits from 10 digits (0-9), here 123 and 231 are same team because all digits 1,2, and 3 belongs to same team such problems are called Combination problems.

Definition 1.11

A combination *is the number of selections of $r \leq n$ different items from n distinguishable items when order of selection is ignored. A combination is denoted by ${}^nC_r$ or, $\binom{n}{k}$, where*

$$^nC_r = \frac{n!}{r!(n-r)!} \tag{1.25}$$

♣

These types of problem generally contains words like table, team, committee, party, etc.

1.6.2 Permutations

Now, let we are interested to sort all 3 digits numbers (repetition is not allowed) from 10 digits (0-9), here 123 and 231 are different numbers, such problems are called Permutation problems.

Note 1:
right-hand
pages, it is
tified.

$y = mx$

Definition 1.12

A permutation *is the number of ways of selecting $r \leq n$ objects from n distinguishable objects when order of selection is important. A permutation is denoted by ${}^nP_r$, where*

$$^nP_r = \frac{n!}{r!} \tag{1.26}$$

♣

These types of problem generally contains words like sort, choose, select, etc.

1.7 The Binomial Theorem

Definition 1.13

The binomial coefficient $\binom{n}{k}$ *is the coefficient of* $a^{n-k}b^k$ *in the expansion of* $(a+b)^n$*, where the expansion is called the* binomial expansion*, written as*

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{k}a^{n-k}b^k + \cdots + b^n \tag{1.27}$$

♣

1.8 Proportional

Definition 1.14

Let two variables are related such that if one of them changes then other variable also changes, then these two variables are said to be proportional *to each other.* ♣

There are two types of proportionality

- Directly proportional
- Inversely proportional

Definition 1.15

Let two variables are proportional to each other such that if one of them increases or decreases than other one also increases or decreases accordingly than they are called directly proportional *to each other.* ♣

Example 1.6 Gravitational force, G between two objects of mass m_1 and m_2 is directly proportional to products of the masses, i.e.

$$G \propto m_1 m_2 \tag{1.28}$$

Definition 1.16

Let two variables are proportional to each other such that if one of them increases or decreases than other one decreases or increases according than they are called inversely proportional *to each other.* ♣

Example 1.7 Gravitational force, G between two objects of distance d is inversely proportional to the square of the distance, i.e.

$$G \propto \frac{1}{d^2} \tag{1.29}$$

Chapter 1 Exercise

1. For practicing problems follow Engineering Mathematics - John Bird- Sixth Edition.

Chapter 2 The Derivative of A Function

Introduction

- *Geometric Interpretation of Derivative*
- *Physical Interpretation of Derivative*
- *Exercise*

2.1 Geometric Interpretation of Derivative

Recap

Equation of a straight line of slope m and cut the y-axis at c is given by

$$y = mx + c. \tag{2.1}$$

Equation of a straight line passing through the points $P(x_1, y_1)$, and $Q(x_2, y_2)$ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1), \tag{2.2}$$

which is also called secant line.

Comparing equations (2.1)-(2.2), can be easily shown that

$$m = \frac{y_2 - y_1}{x_2 - x_1}. \tag{2.3}$$

2.1.1 Tangent Problem

Let $P(x_1, y_1)$, and $Q(x_2, y_2)$ be two points on the curve $y = f(x) = 1/x$, where P is fixed and Q is moving towards P. Then we can write the secant equation PQ,

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1). \tag{2.4}$$

As the point Q approaches P, we say that the secant line PQ is the tangent of y at P.

Definition 2.1 (Tangent)

Let P be a fixed point on a curve $y = f(x)$, and Q be a near by point then the secant line PQ is said to be a tangent line *of y at P, when Q approaches to P, i.e.*

$$\lim_{P\to Q} \text{Secant line } PQ = \text{tangent at } P \text{ of } y. \tag{2.5}$$

♣

From eq. (2.4) the slope of the secant line PQ is

$$m = \frac{y_2 - y_1}{x_2 - x_1} \tag{2.6}$$

Now as $Q \to P$, $x_2 \to x_1$ i.e $\Delta x = x_2 - x_1 \to 0$. Also let $\Delta f = f(x_2) - f(x_1) = y_2 - y_1$, then taking limit on both side of eq. (2.6), we have

$$\lim_{\Delta x\to 0} m = \lim_{\Delta x\to 0} \frac{y_2 - y_1}{x_2 - x_1} = \lim_{\Delta x\to 0} \frac{f(x_2) - f(x_1)}{\Delta x} = \lim_{\Delta x\to 0} \frac{\Delta f}{\Delta x} = \frac{dy}{dx}. \tag{2.7}$$

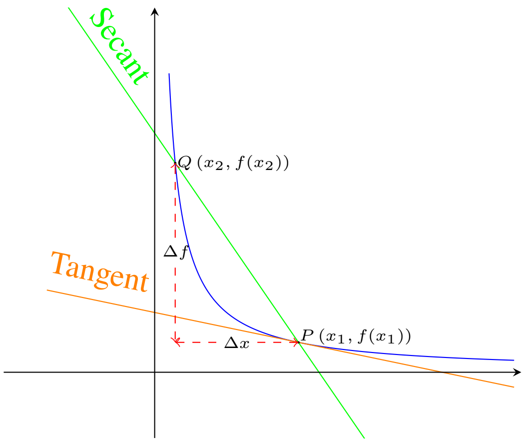


Figure 2.1: Geometric interpretation of derivative.

Definition 2.2 (Derivative)

The derivative of a function $f(x)$ at the point P is the slope of the tangent at the point P. ♣

The basic problem of differential calculus is the problem of tangents: Calculate the slope of the tangent line to the graph at a given point P.

Using eq. (2.7) we can calculate the derivative of y at (x_1, y_1) as

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{\Delta x\to 0} m = \lim_{\Delta x\to 0} \frac{f(x_2)-f(x_1)}{\Delta x} = \lim_{\Delta x\to 0} \frac{1}{\Delta x}\left[\frac{1}{x_1+\Delta x} - \frac{1}{x_1}\right] \\
&= \lim_{\Delta x\to 0} \frac{1}{\Delta x}\left[\frac{x_1 - x_1 - \Delta x}{x_1\,(x_1+\Delta x)}\right] = -\lim_{\Delta x\to 0}\left[\frac{1}{x_1\,(x_1+\Delta x)}\right] \\
\therefore \frac{dy}{dx} &= -\frac{1}{x_1^2}.
\end{aligned}$$

Now the equation of tangent line at $P(x_1, y_1)$ is given by

$$y - y_1 = -\frac{1}{x_1^2}(x - x_1) \tag{2.8}$$

This line cut the Y-axis at

$$x = y_1 x_1^2 + x_1 = 2x_1 \qquad [y = 1/x] \tag{2.9}$$

Similarly, this line cut the X-axis at

$$y = y_1 - \frac{1}{x_1^2}(0 - x_1) = y_1 + y_1 = 2y_1. \qquad [y = 1/x] \tag{2.10}$$

Now area of the triangle form by the tangent and axes is

$$Area = \frac{1}{2}\,(2x_1 - 0)\,(2y_1 - 0) = 2. \tag{2.11}$$

2.2 Physical Interpretation of Derivative

Chapter 2 Exercise

1. Define tangent of a curve.

2. Define derivative in terms of tangent.
3. Show that the area of the triangle form by the tangent of $y = 1/x$ and the axes is 2.
4. Give geometric interpretation of derivative.
5. Give physical interpretation of derivative.

Chapter 3 Limits

Introduction

3.1 Introduction to Limits

Definition 3.1

A constant a is said to be a limit *of the variable x, if*

$$0 < |x - a| < \delta,$$

where δ is pre-assigned positive quantity as small as we please. Symbolically, it is denoted by

$$x \to a \text{ or, } \lim x = a,$$

we say that "x approaches a" or, "x tends to a". ♣

Note *$x \to a$ never implies that $x = a$.*

When x approaches a but always remains smaller than a, we say that x approaches a from the left on the real axis and we write, $x \to a^-$.

Similarly, when x approaches a but always remains greater than a, we say that x approaches a from the right on the real axis and we write, $x \to a^+$.

3.2 Limits of a Function

Definition 3.2

We say that l is the limit of $f(x)$, i.e.

$$\lim_{x \to a} f(x) = l,$$

which means that $f(x)$ is very close to the fixed number l, whenever x is very close to a. If there is no real number l with this property, we say that $f(x)$ has no limit as x approaches a, or that $\lim_{x \to a} f(x)$ does not exist. ♣

This definition does not tell us that how much close $\lim_{x \to a} f(x)$ to l, or, x to a. So, we need more precise definition called (δ, ϵ) definition of limit.

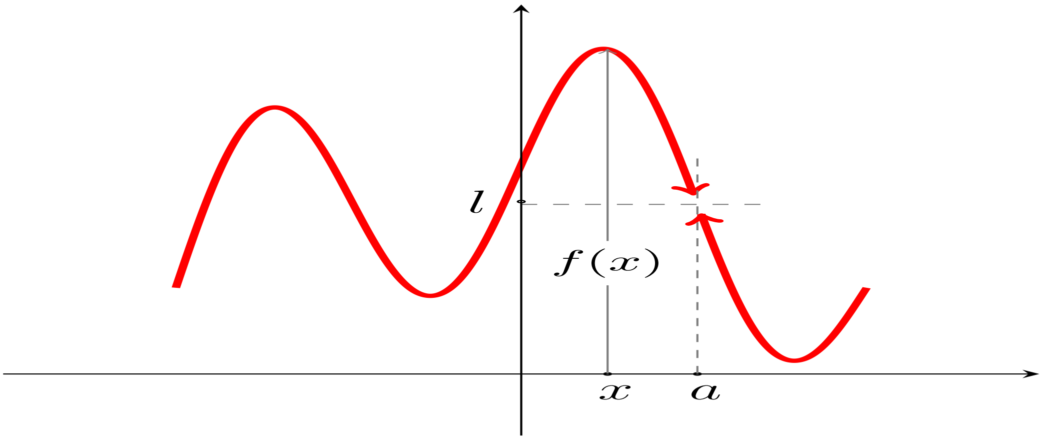


Figure 3.1: As x approaches a, $f(x)$ approaches l.

Definition 3.3 ((δ, ϵ) **definition)**

For each positive number ϵ there exists a positive number δ with the property that

$$|f(x) - l| < \epsilon$$

$\forall x \in D_f$ *that satisfies the inequality*

$$0 < |x - a| < \delta.$$

♣

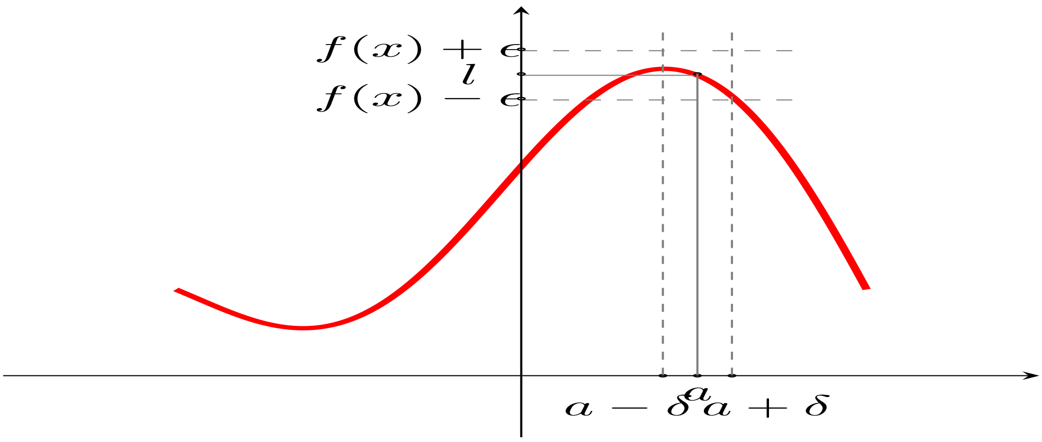


Figure 3.2: As x approaches a, $f(x)$ approaches l.

Note

1. *we are concerned only with the behavior of $f(x)$ near the point $x = a$, and not at all with what happens at $x = a$.*
2. *a may or may not belong to D_f.*

Example 3.1 Prove that

$$\lim_{x \to 2}(3x + 4) = 10$$

by (δ, ϵ) definition of a function.

Solution *Let us consider an arbitrary positive number $\delta > 0$ such that*

$$|3x + 4 - 10| < \epsilon \implies |3x - 6| < \epsilon \implies |x - 2| < \frac{\epsilon}{3} \implies |x - 2| < \delta,$$

where $\delta = \frac{\epsilon}{3}$. *That is*

$$\lim_{x \to 2}(3x+4) = 10.$$

Example 3.2 Prove that

$$\lim_{x \to a} \frac{x^2 - a^2}{x - a} = 2a$$

by (δ, ϵ) definition of a function.

Solution *Let us consider an arbitrary positive number* $\delta > 0$ *such that*

$$\begin{aligned} & \left|\frac{x^2 - a^2}{x - a} - 2a\right| < \epsilon \\ \implies & \left|\frac{(x+a)(x-a)}{x-a} - 2a\right| < \epsilon \\ \implies & |x + a - 2a| < \epsilon \\ \implies & |x - a| < \delta, \end{aligned}$$

where $\delta = \epsilon$. *That is*

$$\lim_{x \to a} \frac{x^2 - a^2}{x - a} = 2a.$$

The definition 3.2 can also be written as follows

Definition 3.4

We say that l *is the limit of* $f(x)$, *i.e.*

$$\lim_{h \to 0} f(x+h) = l,$$

which means that $f(x+h)$ *is very close to the fixed number* l, *whenever* h *is very close to* 0. *If there is no real number* l *with this property, we say that* $f(x+h)$ *has no limit as* h *approaches* 0, *or that* $\lim_{h \to 0} f(x+h)$ *does not exist.*

♣

Definition 3.5

Let f *be a function and* $S \subset D_f$ *such that any number* $x \in S$ *is less than* a *then for each positive number* ϵ *there exists a positive number* δ *with the property that*

$$|f(x) - l_1| < \epsilon$$

$\forall x \in S$ *that satisfies the inequality*

$$0 < |x - a| < \delta.$$

We say that l_1 *is the* left hand limit *of* $f(x)$ *as* $x \to a$ *and we write*

$$\lim_{x \to a^-} f(x) = l_1,$$

♣

Definition 3.6

Let f *be a function and* $S \subset D_f$ *such that any number* $x \in S$ *is greater than* a *then for each positive number* ϵ *there exists a positive number* δ *with the property that*

$$|f(x) - l_2| < \epsilon$$

$\forall x \in S$ that satisfies the inequality

$$0 < |x - a| < \delta.$$

We say that l_2 is the right hand limit *of $f(x)$ as $x \to a$ and we write*

$$\lim_{x \to a^+} f(x) = l_2,$$

♣

When both left and right hand limit of $f(x)$ exist at a, we say that limit of $f(x)$ exist at a. Moreover, if $l_1 = l_2 = l$ then it is equal to l. When $\lim_{x \to a} f(x) = f(a)$ then the function is said to be continuous at a.

Example 3.3 Investigate the function

$$f(x) = \begin{cases} 1 + 2x, & -\frac{1}{2} \leq x < 0, \\ 1 - 2x, & 0 \leq x < \frac{1}{2}, \\ -1 + 2x, & x > \frac{1}{2}, \end{cases}$$

at $x = 0$ and $x = \frac{1}{2}$.

Solution *For $x = 0$,*

$$\lim_{h \to 0^-} f(0 + h) = \lim_{h \to 0^-} 1 + 2h = 1.$$

$$\lim_{h \to 0^+} f(0 + h) = \lim_{h \to 0^+} 1 - 2h = 1.$$

and $f(0) = 1 - 2 \cdot 0 = 1$. That is

$$\lim_{h \to 0} f(0) = 1 = f(0).$$

Hence, $f(x)$ is continuous.

For $x = \frac{1}{2}$,

$$\lim_{h \to 0^-} f\left(\frac{1}{2} + h\right) = \lim_{h \to 0^-} 1 - 2\left(\frac{1}{2} + h\right) = 0.$$

$$\lim_{h \to 0^+} f\left(\frac{1}{2} + h\right) = \lim_{h \to 0^-} -1 + 2\left(\frac{1}{2} + h\right) = 0.$$

and

$$\lim_{h \to 0^-} f\left(\frac{1}{2} + h\right) = \lim_{h \to 0^+} f\left(\frac{1}{2} + h\right) = 0.$$

That is

$$\lim_{h \to 0} f(0) = 0.$$

Example 3.4 Evaluate

$$\lim_{x \to 0} \frac{x}{|x|}.$$

Solution

$$\lim_{h \to 0^-} f(x - h) = \lim_{h \to 0^-} \frac{x}{-x} = -1.$$

$$\lim_{h \to 0^+} f(x + h) = \lim_{h \to 0^+} \frac{x}{x} = 1.$$

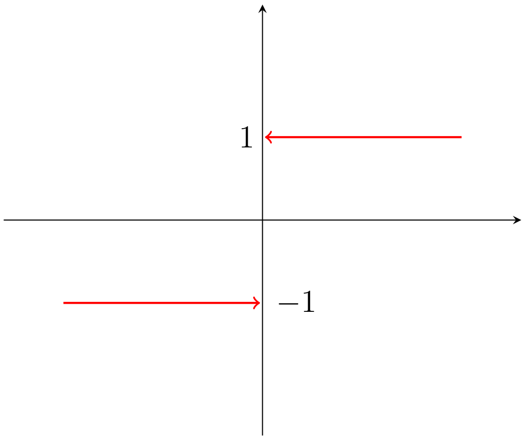


Figure 3.3: Limit of $x/|x|$ at 0.

As it found that $\lim_{h\to 0^-} f(x-h) \neq \lim_{h\to 0^+} f(x+h)$ *and also evident from the* ***Figure*** *3.3. Hence, it does not have any limit at* $x = 0$*.*

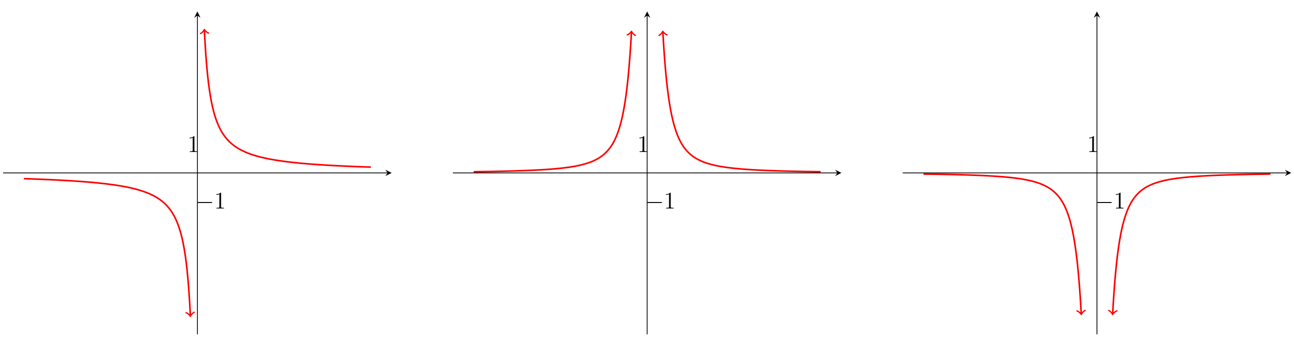
3.3 Infinite Limits

If a variable x assumes all positive (negative) values and increases (decreases) without limit such that it is greater (less) than any positive (negative) number, however big (small) which we may imagine, x is said to tends to infinity (minus infinity) and symbolically it is written as $x \to -\infty$.

Example 3.5 Evaluate

(a) $\lim_{x\to 0} \frac{1}{x}$,
(b) $\lim_{x\to 0} \frac{1}{x^2}$,
(c) $\lim_{x\to 0} -\frac{1}{x^2}$.

Solution



(a) Limit of $1/x$ at 0. (b) Limit of $1/x^2$ at 0. (c) Limit of $-1/x^2$ at 0.

Figure 3.4: Infinite limits.

(a) Let $f(x) = \frac{1}{x}$.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1}{x} = \infty.$$

As it found that $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$ *and also evident from the* ***Figure*** *3.4a. Hence, it does not have any limit at* $x = 0$.

(b) Let $f(x) = \frac{1}{x^2}$.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{x^2} = \infty.$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1}{x^2} = \infty.$$

As it found that $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = \infty$ *and also evident from the* ***Figure*** *3.4b. Therefore limit exists and* $\lim_{x\to 0} \frac{1}{x^2} = \infty$.

(c) Let $f(x) = -\frac{1}{x^2}$.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} -\frac{1}{x^2} = -\infty.$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} -\frac{1}{x^2} = -\infty.$$

As it found that $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = -\infty$ *and also evident from the* ***Figure*** *3.4c. Therefore limit exists and* $\lim_{x\to 0} -\frac{1}{x^2} = -\infty$.

Example 3.6 Evaluate

$$\lim_{x\to 2} \frac{3}{(x-2)^2}.$$

Solution

$$\lim_{x\to 2} \frac{3}{(x-2)^2} = 3\lim_{h\to 0} \frac{1}{(2+h-2)^2} = 3\lim_{h\to 0} \frac{1}{h^2} = \infty. \qquad \text{[Using Ex. 3.5b.]}$$

3.4 Limits at Infinity

Let a be any positive numbers, that $f(x)$ is defined for all numbers $x \geq a$. We say that $f(x)$ approaches l as x tends to ∞, and we write

$$\lim_{x\to\infty} f(x) = l.$$

If the following condition is satisfied. Given any $\epsilon > 0$, there exists a positive number C, such that whenever $x > C$, we have

$$|f(x) - l| < \epsilon.$$

Again, let a be any negative numbers, that $f(x)$ is defined for all numbers $x \leq a$. We say that $f(x)$ approaches l as x tends to $-\infty$, and we write

$$\lim_{x\to-\infty} f(x) = l.$$

If the following condition is satisfied. Given any $\epsilon > 0$, there exists a positive number C, such that whenever $x < -C$, we have

$$|f(x) - l| < \epsilon.$$

Example 3.7 Evaluate

$$\lim_{n\to 0} \frac{n^2+n-1}{3n^2+1}.$$

Solution

$$\lim_{n\to\infty}\frac{n^2+n-1}{3n^2+1}=\lim_{n\to\infty}\frac{1+\frac{1}{n}-\frac{1}{n^2}}{3+\frac{1}{n^2}}=\frac{1}{3}.$$

Example 3.8 Show that

$$\lim_{x\to\infty}\frac{\sin x}{x}=0. \tag{3.1}$$

Solution *We have,* $|\sin x|\leq 1$, *for any values of* x. *Hence,*

$$\left|\frac{\sin x}{x}\right| \tag{3.2}$$

3.5 Fundamental Theorems of Limits

Theorem 3.1

$$\lim_{x\to a} x = a.$$

Corollary 3.1

If c *is a constant then*

$$\lim_{x\to a} c = c.$$

Theorem 3.2

$$\lim_{x\to a}\{f(x)\pm g(x)\}=\lim_{x\to a} f(x)\pm\lim_{x\to a} g(x).$$

Theorem 3.3

$$\lim_{x\to a}\{f(x)g(x)\}=\lim_{x\to a} f(x)\lim_{x\to a} g(x).$$

Corollary 3.2

If c *is a constant then*

$$\lim_{x\to a} cf(x)=c\lim_{x\to a} f(x).$$

Theorem 3.4

If $\lim_{x\to a} f(x)\neq 0$ *then*

$$\lim_{x\to a}\frac{1}{f(x)}=\frac{1}{\lim_{x\to a} f(x)}.$$

Corollary 3.3

If $\lim_{x \to a} g(x) \neq 0$ *then*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}.$$

Theorem 3.5

If

$$\lim_{x \to a} f(x) = t \text{ and } \lim_{u \to t} g(u) = g(t)$$

i.e. if $g(u)$ *is contentious at* $u = t$*, then*

$$\lim_{x \to a} g\left(f(x)\right) = g\left(\lim_{x \to a} f(x)\right).$$

Example 3.9 If

$$\lim_{x \to a} f(x) = l,$$

then

$$\lim_{x \to a} \sin\left(f(x)\right) = \sin\left(\lim_{x \to a} f(x)\right) = \sin l.$$

$$\lim_{x \to a} e^{(f(x))} = e^{\left(\lim_{x \to a} f(x)\right)} = e^{l}.$$

$$\lim_{x \to a} \left(f(x)\right)^n = \left(\lim_{x \to a} f(x)\right)^n = l^n.$$

Definition 3.7 (Limits of a Sequence)

A number l *is the limit of a sequence* $a_1, a_2, \ldots, a_n$ *or,*

$$\lim_{x \to \infty} a_n = l,$$

if for every $\epsilon > 0$*, there is a number* N *such that*

$$|a_n - l| < \epsilon$$

when $n \geq N$*.*

3.6 Some Important Limits

Example 3.10 Prove that

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1.$$

Solution

Let the tangents at P *and* Q *meets at* T*.* PQ *is a chord of the circle and* R *is the middle point. So, we have,*

$$\angle ROQ = \angle ROP = \theta.$$

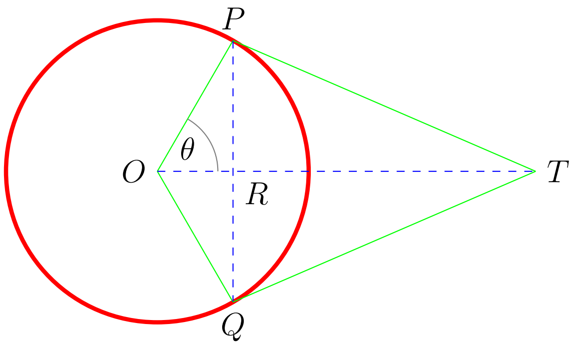


Figure 3.5: Limit of $\frac{\sin\theta}{\theta}$ at 0.

Also,

$$\angle TPR = \angle TQR = \theta.$$

Now, we have,

$$\begin{aligned} & PQ < \overset{\frown}{PQ} < PT + QT \\ \implies & 2PR < 2\theta < 2PT \\ \implies & PR < \theta < PT \\ \implies & \frac{PR}{OP} < \frac{\theta}{OP} < \frac{PT}{OP} \\ \implies & \sin\theta < \theta < \tan\theta \\ \implies & 1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta} \\ \implies & \cos\theta < \frac{\sin\theta}{\theta} < 1 \\ \implies & \lim_{\theta\to 0}\cos\theta < \lim_{\theta\to 0}\frac{\sin\theta}{\theta} < \lim_{\theta\to 0} 1 \\ \implies & 1 < \lim_{\theta\to 0}\frac{\sin\theta}{\theta} < 1 \end{aligned}$$

Hence,

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.$$

Example 3.11 Prove that

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$$

Solution *Expanding binomially,*

$$(1+x)^{\frac{1}{x}} = 1 + \frac{1}{x}x + \frac{(1/x)(1/x-1)}{2!}x^2 + \cdots + = 1 + 1 + \frac{(1-x)}{2!} + \frac{(1-x)(1-2x)}{3!} + \ldots$$

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots = e.$$

Example 3.12 Prove that

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e \tag{3.3}$$

Solution *Expanding binomially,*

$$\left(1+\frac{1}{x}\right)^x = 1 + x\frac{1}{x} + \frac{x(x-1)}{2!}\frac{1}{x^2} + \cdots + = 1 + 1 + \frac{(1-1/x)}{2!} + \frac{(1-1/x)(1-2/x)}{3!} + \ldots$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots = e.$$

Example 3.13 Prove that

$$\lim_{x\to 0}\left\{\frac{1}{x}\ln(1+x)\right\} = 1$$

Solution

$$\lim_{x\to 0}\left\{\frac{1}{x}\ln(1+x)\right\} = \lim_{x\to 0}\left\{\ln(1+x)^{\frac{1}{x}}\right\} = \ln\left\{\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right\} = \ln e = 1.$$

Example 3.14 Prove that

$$\lim_{x\to 0}\frac{a^x-1}{x} = \ln a$$

Solution *Let*

$$\begin{aligned} & a^x - 1 = y \\ \implies\quad & x\ln a = \ln(1+y) \\ \implies\quad & x = \frac{\ln(1+y)}{\ln a}. \end{aligned}$$

Now,

$$\frac{a^x-1}{x} = \frac{y\ln a}{\ln(1+y)} = \frac{\ln a}{1/y\,\ln(1+y)} = \frac{\ln a}{\ln(1+y)^{1/y}}$$

Taking limit,

$$\lim_{x\to 0}\frac{a^x-1}{x} = \lim_{y\to 0}\frac{\ln a}{\ln(1+y)^{1/y}} = \ln a\lim_{y\to 0}\frac{1}{\ln(1+y)^{1/y}} = \ln a.$$

Example 3.15 Prove that

$$\lim_{x\to 0}\frac{e^x-1}{x} = 1$$

Solution *Similar as example 3.14.*

Example 3.16 Prove that

$$\lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1},$$

for all rational values of n, provided a is positive.

Solution *Case I: When n is positive integer.*

$$\begin{aligned}\lim_{x\to a}\frac{x^n-a^n}{x-a} &= \lim_{x\to a}\frac{(x-a)\left(x^{n-1}+x^{n-2}a+\cdots+a^{n-1}\right)}{x-a} \\ &= \lim_{x\to a}\left(x^{n-1}+x^{n-2}a+\cdots+a^{n-1}\right) = na^{n-1}. \end{aligned} \tag{3.4}$$

Case II: When n is negative integer. Let $n = -m$, where m being a positive integer and $a \neq 0$.

$$\begin{aligned}
\lim_{x\to a} \frac{x^n - a^n}{x - a} &= \lim_{x\to a} \frac{x^{-m} - a^{-m}}{x - a} \\
&= \lim_{x\to a} \frac{1}{x-a}\left(\frac{1}{x^m} - \frac{1}{a^m}\right) \\
&= \lim_{x\to a} \frac{-1}{a^m x^m}\left(\frac{x^m - a^m}{x - a}\right) \\
&= \lim_{x\to a} \frac{-1}{a^m x^m} \lim_{x\to a}\left(\frac{x^m - a^m}{x - a}\right) \\
&= \frac{-1}{a^{2m}} m a^{m-1} \qquad \text{[Using (3.4).]} \\
&= (-m)a^{(-m)-1} = na^{n-1}.
\end{aligned}$$

Example 3.17 Prove that

$$\lim_{x\to 0} \frac{(1+x)^n - 1}{x} = n.$$

Solution

$$\begin{aligned}
& \lim_{x\to 0} \frac{(1+x)^n - 1}{x} \\
\Longrightarrow \quad & \lim_{x\to 0} \frac{1 + nx + x^2\frac{n(n-1)}{2!} + \cdots - 1}{x} = n
\end{aligned}$$

3.7 How to Find Limits?

3.7.1 Easy Limits

Example 3.18

$$\lim_{x\to 0} 4x^2 + 7 = 7.$$

3.7.2 Limits of the form $0/0$

Example 3.19

$$\lim_{x\to a} \frac{x^4 - a^4}{x - a} = \lim_{x\to a} \frac{(x-a)(x^3 + ax^2 + a^2x + a^3)}{x - a} = \lim_{x\to a}(x^3 + ax^2 + a^2x + a^3) = 4a^3.$$

Example 3.20

$$\lim_{x\to 0} \frac{\sqrt{1+x} - 1}{\sqrt{x}} = \lim_{x\to 0} \frac{\left(\sqrt{1+x} - 1\right)\left(\sqrt{1+x} + 1\right)}{\sqrt{x}\left(\sqrt{1+x} + 1\right)} = \lim_{x\to 0} \frac{1 + x - 1}{\sqrt{x}\left(\sqrt{1+x} + 1\right)} = \lim_{x\to 0} \frac{\sqrt{x}}{\left(\sqrt{1+x} + 1\right)} = 0.$$

Example 3.21

$$\lim_{x\to 0} \frac{1 - \cos x}{x} = \lim_{x\to 0} \frac{2\sin^2\frac{x}{2}}{x} == \lim_{x\to 0} \sin\frac{x}{2} \lim_{x\to 0} \frac{\sin\frac{x}{2}}{\frac{x}{2}} = 0.$$

3.7.3 Limits of the form ∞/∞

Example 3.22

$$\lim_{n\to\infty}\frac{n^2+n-1}{3n^2+1}=\lim_{n\to\infty}\frac{1+\frac{1}{n}-\frac{1}{n^2}}{3+\frac{1}{n^2}}=\frac{1}{3}.$$

Example 3.23

$$\lim_{x\to\infty}\frac{\sin x}{x}=\lim_{x\to\infty}\sin x\lim_{x\to\infty}\frac{1}{x}=0.$$

Example 3.24

$$\lim_{x\to 0}\cos\frac{1}{x}=\cos\left(\lim_{x\to 0}\frac{1}{x}\right).$$

Example 3.25

$$\lim_{x\to\infty}\frac{3^x-3^{-x}}{3^x+3^{-x}}=\lim_{x\to\infty}\frac{\frac{3^x-3^{-x}}{3^x}}{\frac{3^x+3^{-x}}{3^x}}=\lim_{x\to\infty}\frac{1-3^{-2x}}{1+3^{-2x}}=1.$$

3.7.4 L'Hôpital's Rule

Indeterminate forms, such as $\frac{0}{0}$, $\frac{\infty}{\infty}$, 1^∞, 0^0, ∞^0, $0\cdot\infty$, and $\infty-\infty$, can sometimes be evaluated using l'Hôpital's (pronounce as Lopital) rule.

Definition 3.8

L'Hôpital's rule states that for functions $f(x)$ and $g(x)$ which are differentiable on an open interval I except possibly at a point c contained in I, if $\lim_{x\to c} f(x)=\lim_{x\to c} g(x)=0$, or, $\pm\infty$, and $g'(x)\neq 0$, for all $x\in I$ with $x\neq c$, and $\lim_{x\to c}\frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}.$$

♣

Example 3.26 Evaluate

$$\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)$$

Solution

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right) &= \lim_{x\to 0}\frac{\sin^2 x-x^2}{x^2\sin^2 x}\\
&= \lim_{x\to 0}\frac{2\sin x\cos x-2x}{2x\sin^2 x+2x^2\sin x\cos x} && \text{[Applying L'Hôpital's rule]}\\
&= \lim_{x\to 0}\frac{\sin 2x-2x}{x\left(1-\cos 2x+x\sin 2x\right)}\\
&= \lim_{x\to 0}\frac{2\left(\cos 2x-1\right)}{\left(1-\cos 2x+x\sin 2x\right)+x\left(3\sin 2x+2x\cos 2x\right)} && \text{[Applying L'Hôpital's rule]}\\
&= \lim_{x\to 0}\frac{-4\sin 2x}{2\left(3\sin 2x+2x\cos 2x\right)+4x\left(2\cos 2x-x\sin 2x\right)} && \text{[Applying L'Hôpital's rule]}\\
&= \lim_{x\to 0}\frac{-2\sin 2x}{\left(3\sin 2x+2x\cos 2x\right)+2x\left(2\cos 2x-x\sin 2x\right)}
\end{aligned}$$

$$= \lim_{x\to 0} \frac{-4\cos 2x}{6\left(2\cos 2x - x\sin 2x\right) + 4x\left(-3\sin 2x + x\cos 2x\right)} \quad \text{[Applying L'Hôpital's rule]}$$

$$= \frac{-4}{6(2-0)+0} = -\frac{1}{3}$$

3.8 Application of Limits

Chapter 3 Exercise

1. Define following
   (a). Limit of a constant.
   (b). Limit of a function.
   (c). (δ, ϵ) definition of limit of a function.
   (d). Right hand and left hand limit.
   (e). L'Hôpital's rule.
2. When a function has a limit?
3. Can a function has two limits at a single point within a given interval? or at two points?
4. By (δ, ϵ) definition prove that

$$\lim_{x\to a} \frac{x^2 - a^2}{x - a} = 2a.$$

5. By (δ, ϵ) definition prove that

$$\lim_{x\to 2} x^3 - 3x + 7 = 9.$$

6. Show that
   (a). $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{\sqrt{x}} = 0$
   (b). $\lim_{x\to a} \frac{x^4-a^4}{x-a} = 4a^3$
   (c). $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$
   (d). $\lim_{n\to\infty} \frac{n^2+n-1}{3n^2+1} = \frac{1}{3}$
   (e). $\lim_{x\to 0} \frac{\tan x}{x} = 1$
   (f). $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} = \frac{1}{2}$
   (g). $\frac{\sin 5x}{\sin 6x} = \frac{5}{6}$
7. Find the following limits
   (a). $\lim_{x\to\infty} \frac{3^x-3^{-x}}{3^x+3^{-x}}$, Ans. 1.
   (b). $\lim_{x\to 0} \cos\frac{1}{x}$, Ans. Does not exist.
   (c). $\lim_{x\to\infty} \frac{\sin x}{x}$, Ans. 0.
8. Using L'Hôpital's rule find the following limits
   (a). $\lim_{x\to 0} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right)$ Ans. $-\frac{1}{3}$.
   (b). $\lim_{\theta\to\frac{\pi}{4}} \frac{\sqrt{2}-\cos\theta+\sin\theta}{(4\theta-\pi)^2}$ Ans. 0.
   (c). $\lim_{x\to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$, Ans. $\frac{1}{2}$.

Chapter 4 Continuity

Introduction

- *Continuity*
- *Cauchy's Definition of Continuity*
- *Classification of Discontinuity*
- *Properties of Continuous Functions*
- *Continuity of Some Elementary Functions*
- *Differentiability of a Function*
- *Exercise*

4.1 Continuity

Definition 4.1

A function $f(x)$ is said to be continuous *at $x = a$, if f is defined at $x = a$ and $\lim_{x\to a} f(x) = f(a)$. If any of the above conditions is not satisfied then $f(x)$ is said to have a* discontinuity *at $x = a$.* ♣

Example 4.1 Prove that the function $f(x) = \sin x$ is continuous for every values of x.

Solution $\sin x$ *is defined for all values of x. Therefore, $D_f = \mathbb{R} = (-\infty, \infty)$. For any $x \in \mathbb{R}$,*

$$\begin{aligned}\lim_{h\to 0} f(x+h) &= \lim_{h\to 0} \sin(x+h) = \lim_{h\to 0} (\sin x \cos h + \cos x \sin h) \\ &= \sin x \lim_{h\to 0} \cos h + \cos x \lim_{h\to 0} \sin h = \sin x = f(x).\end{aligned}$$

Hence, $f(x) = \sin x$ is continuous for all real values of x.

Example 4.2 Discuss the continuity of the function $f(x) = \frac{1}{3-e^{\frac{1}{x}}}$ at $x = 0$.

Solution *Since $\frac{1}{0}$ is undefined, therefore $f(0)$ is undefined. Hence $f(x)$ is discontinuous at $x = 0$.*

4.2 Cauchy's Definition of Continuity

Definition 4.2

A function $f(x)$ is continuous *at $x = a$, if $f(a)$ is defined and for a small positive number ϵ there is a positive number δ can be always determined such that*

$$|f(x) - f(a)| < \epsilon$$

where

$$|x - a| \le \delta.$$

♣

4.3 Classification of Discontinuity

A function $f(x)$ is said to be discontinuous for $x = a$ if $f(a)$ is not defined or $\lim x \to 0 f(x)$ does not exist.

The different type of discontinuous are

1. Ordinary discontinuity or, discontinuity of the first kind,
2. Discontinuity of the second kind,
3. Mixed discontinuity,
4. Removal discontinuity,
5. Infinite discontinuity, and
6. Oscillatory discontinuity.

4.3.1 Ordinary Discontinuity or, Discontinuity of the First Kind

Definition 4.3
If the function $f(x)$ has finite limit but

$$\lim_{h\to 0^-} f(a-h) \neq \lim_{h\to 0^+} f(a-h) \neq f(a)$$

then the function is said to have ordinary discontinuity or, discontinuity of the first kind *at $x = a$.* ♣

4.3.2 Discontinuity of the Second Kind

Definition 4.4
If the limits of $f(x)$, $\lim_{h\to 0^-} f(a-h)$ and $\lim_{h\to 0^+} f(a+h)$ do not exist for $x = a$ then the function is said to have discontinuity of the second kind *at $x = a$.* ♣

4.3.3 Mixed Discontinuity

Definition 4.5
If one of the limits of $f(x)$ exists then the discontinuities of the function $f(x)$ at $x = a$ is called mixed discontinuity. ♣

That is if $\lim_{h\to 0^-} f(a-h) = f(a)$ but $\lim_{h\to 0^+} f(a+h) \neq f(a)$, then $f(x)$ is continuous on the right but it has a ordinary discontinuity at the left for $x = a$.

Similarly, if $\lim_{h\to 0^+} f(a+h) = f(a)$ but $\lim_{h\to 0^-} f(a-h) \neq f(a)$, then $f(x)$ is continuous on the left but it has a ordinary discontinuity at the right for $x = a$.

Example 4.3 Examine the continuity of the function $f(x)$ at $x = \frac{3}{2}$, where

$$f(x) = \begin{cases} 3 - 2x & 0 \leq x < \frac{3}{2} \\ -3 - 2x & x \geq \frac{3}{2}. \end{cases}$$

Solution *Clearly, $\frac{3}{2} \in D_f$ and $f(\frac{3}{2}) = -6$.*

Again,

$$\lim_{h\to 0^+} f(\frac{3}{2}+h) = \lim_{h\to 0^+}\left(-3-2\left(\frac{3}{2}+h\right)\right) = -6$$

$$\lim_{h\to 0^-} f(\frac{3}{2}-h) = \lim_{h\to 0^-}\left(3-2\left(\frac{3}{2}-h\right)\right) = 0.$$

$$\therefore \lim_{h\to 0^+} f\left(\frac{3}{2}+h\right) = f\left(\frac{3}{2}\right) \neq \lim_{h\to 0^-} f\left(\frac{3}{2}-h\right)$$

Hence, $f(x)$ has a discontinuity at left for $x = \frac{3}{2}$.

4.3.4 Removal Discontinuity

Definition 4.6

If $\lim_{h\to 0^-} f(a-h) = \lim_{h\to 0^+} f(a+h) \neq f(a)$ *then the function* $f(x)$ *is said to have a* removable discontinuity *for* $x = a$*.* ♣

Example 4.4 Show that the function

$$f(x) = \begin{cases} \frac{x^2-a^2}{x-a}, & x \neq a, \\ 0, & x = a, \end{cases}$$

has a removable discontinuity at $x = a$.

Solution *We have* $a \in D_f$ *and* $f(a) = 0$.

Again

$$\lim_{h\to 0^-} f(a-h) = \lim_{h\to 0^-} \frac{(a-h)^2 - a^2}{a-h-a} = \lim_{h\to 0^-} \frac{-2ah+h^2}{-h} = 2a$$

$$\lim_{h\to 0^+} f(a+h) = \lim_{h\to 0^+} \frac{(a+h)^2 - a^2}{a+h-a} = \lim_{h\to 0^+} \frac{2ah+h^2}{h} = 2a$$

$$\therefore \lim_{h\to 0^-} f(a-h) = \lim_{h\to 0^+} f(a+h) \neq f(a).$$

Hence, $f(x)$ is discontinuous at $x = a$, and $f(x)$ has a removable discontinuity at $x = a$.

4.3.5 Infinite Discontinuity

Definition 4.7

If both the limits of $f(x)$ *are infinite, then the function* $f(x)$ *has an* infinite discontinuity *at* $x = a$*.* ♣

If $\lim_{h\to 0^-} f(a-h)$ and $\lim_{h\to 0^+} f(a+h)$ tend to $\pm\infty$, then the function $f(x)$ has an infinite discontinuity at $x = a$.

Example 4.5 Show that the function

$$f(x) = \frac{x-2}{x-1}$$

has an infinite discontinuity at $x = 1$.

Solution *Since* $f(1) = \frac{-1}{0}$ *is undefined,* $1 \notin D_f$*, therefore* $f(x)$ *is discontinuous at* $x = 1$*.*

Now

$$\lim_{h\to 0}\left|\frac{-1+h}{h}-\frac{-1-h}{-h}\right| = \lim_{h\to 0}\left|-\frac{2}{h}\right| = \infty.$$

Hence, the discontinuity at $x = 1$ is infinite.

4.3.6 Oscillatory Discontinuity

Definition 4.8

A function $f(x)$ having a discontinuity at a point $x = a$ may oscillate finitely or does not tend to finite limit or to $\pm\infty$ as x tends to ∞, we say that $f(x)$ has an oscillatory discontinuity *at $x = a$.* ♣

Example 4.6 Discuss the continuity of $\sin\left(\frac{1}{x}\right)$.

Solution $\sin\left(\frac{1}{x}\right)$ *oscillates between -1 and 1 and more rapidly as x approaches 0 from either sides. $f(x)$ oscillates finitely at $x = 0$.*

4.4 Properties of Continuous Functions

1. The sum or difference of two continuous function is a continuous function over the intersection at their domain.
2. The product of two continuous function is a continuous function over the intersection at their domain.
3. The quotient of two continuous function is a continuous function over the intersection at their domain, if the denominator is not zero anywhere in it.
4. If a function is continuous in a closed interval, it is bounded in the interval.
5. A function which is continuous in a closed interval attains at least once its least upper and greatest lower bound.
6. A continuous function which has opposite sign at two points meets its domain vanishes at least one between these points.
7. A continuous function $f(x)$, in the interval (a, b), assumes at least once every values between $f(a)$ and $f(b)$, it being supposed that $f(a) \neq f(b)$.
8. The converse of this theorem is not true i.e, a function $f(x)$ which takes all values between $f(a)$ and $f(b)$ is not necessarily continuous in the interval (a, b).

4.5 Continuity of Some Elementary Functions

Theorem 4.1

The function

$$f(x) = x^n$$

is continuous for all values of x, when n is any rational number, except at $x = 0$, when n is negative. ♡

Proof Let us investigate the continuity of the function at $x = a$.

$$\lim_{h\to 0^+} f(a+h) = \lim_{h\to 0^+} (a+h)^n = \lim_{h\to 0^+} a^n(1 + n\left(\frac{h}{a}\right) + \dots) = a^n.$$

$$\lim_{h\to 0^-} f(x-h) = \lim_{h\to 0^-} (a-h)^n = \lim_{h\to 0^-} a^n(1 - n\left(\frac{h}{a}\right) + \dots) = a^n.$$

When n is negative then let $n = -m$, where m is positive. Then $x^n = x^{-m}$, which is undefined for $x = 0$.

Hence, $f(x) = x^n$ is continuous for all values of x, except $x = 0$, when n is negative.

Corollary 4.1

Polynomials are continuous functions. ♡

Theorem 4.2

Rational algebraic functions $R(x) = \frac{P(x)}{Q(x)}$, where $Q(x) \neq 0$ is continuous functions if $P(x)$ and $Q(x)$ are continuous for all values of x. ♡

Theorem 4.3

Exponential function e^x is continuous for all real values of x. ♡

Theorem 4.4

Logarithm function $\ln x$ is continuous for all positive values of x, i.e. $x > 0$. ♡

4.6 Differentiablity of a Function

Definition 4.9

A function $f(x)$ is said to be differentiable at $x = a$, if $a + h$, and a belong to the domain of f as $h \to 0$, and

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

exists, we write

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h},$$

provided the limit exists. ♣

If
$$\lim_{h^+\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0^-}\frac{f(a-h)-f(a)}{-h}$$
or,
$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$
exists, then the function $f(x)$ is differentiable at $x=a$.

Theorem 4.5

Every finitely derivable function is continuous. ♡

Proof Let $f(x)$ be differentiable at $x=a$ i.e,
$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$
$$\lim_{h\to 0}(f(a+h)-f(a))=\lim_{h\to 0}hf'(a)=0$$
$$\lim_{h\to 0}f(a+h)=f(a).$$
Hence, $f(x)$ is continuous at $x=a$.

The converse of this theorem is not necessarily is true, i.e. a function may be continuous for a value of the variable in an interval but derivative at this point may not exist.

Example 4.7 Consider the function
$$f(x)=\begin{cases}x, & 0\le x<\frac{1}{2}\\ (1-x), & \frac{1}{2}\le x<1.\end{cases}$$
Is the function continuous at $x=\frac{1}{2}$? Is it differentiable at $x=\frac{1}{2}$? Draw the graph.

Solution *$x=\frac{1}{2}\in D_f$, also $f\left(\frac{1}{2}\right)=1-\frac{1}{2}=\frac{1}{2}$.*
$$\lim_{h^0-}f\left(\tfrac{1}{2}-h\right)=\lim_{h^0-}\left(\tfrac{1}{2}-h\right)=\tfrac{1}{2},$$
$$\lim_{h^0+}f\left(\tfrac{1}{2}+h\right)=\lim_{h^0+}\left(\tfrac{1}{2}+h\right)=\tfrac{1}{2}.$$
Thus
$$\lim_{h^0-}f\left(\frac{1}{2}-h\right)=\lim_{h^0+}f\left(\frac{1}{2}+h\right)=f\left(\frac{1}{2}\right).$$

Hence the function $f(x)$ is continuous at $x=\frac{1}{2}$.

Again
$$\lim_{h\to 0^-}\frac{f\left(\frac{1}{2}-h\right)-f\left(\frac{1}{2}\right)}{-h}=\lim_{h\to 0^-}\frac{\frac{1}{2}-h-\frac{1}{2}}{-h}=1,$$
$$\lim_{h\to 0^+}\frac{f\left(\frac{1}{2}+h\right)-f\left(\frac{1}{2}\right)}{h}=\lim_{h\to 0^+}\frac{1-\frac{1}{2}-h-\frac{1}{2}}{h}=-1,$$
Thus right hand limit and left hand limit are not equal. Hence $f'(\frac{1}{2})$ does not exit, i.e. $f(x)$ is not

differentiable at $x = \frac{1}{2}$.

Example 4.8 A function $f(x)$ is defined in the following way.

$$f(x) = \begin{cases} 0, & 0 \leq x < 3, \\ 4 & x = 3, \\ 5 & 3 < x \leq 4 \end{cases}$$

Investigate the continuity and differentiability at $x = 3$

Solution $x = 3 \in D_f$, *also* $f(3) = 4$.

$$\lim_{h^0-} f(3-h) = 0,$$
$$\lim_{h^0+} f(3+h) = 5.$$

Thus

$$\lim_{h \to 0^-} f(3-h) \neq \lim_{h \to 0^+} f(3+h) \neq f(3).$$

Hence the function $f(x)$ *is discontinuous at* $x = 3$.

Again

$$\lim_{h \to 0^-} \frac{f(3-h)-f(3)}{-h} = \lim_{h \to 0^-} \frac{0-4}{-h} = \infty,$$
$$\lim_{h \to 0^+} \frac{f(3+h)-f(3)}{h} = \lim_{h \to 0^+} \frac{5-4}{h} = \infty,$$

$Lf'(3) \neq Rf'(3)$, *that is* $f'(3)$ *does not exists.*

Hence $f'(3)$ *is not differentiable at* $x = 3$.

Example 4.9 If

$$f(x) = \begin{cases} 1, & x < 0, \\ 1 + \sin x, & 0 \leq x < \frac{\pi}{2} \\ 2 + \left(x - \frac{\pi}{2}\right)^2, & x \geq \frac{\pi}{2} \end{cases}$$

Discuss the continuity and differentiability of the function at $x = \frac{\pi}{2}$.

Solution $x = \frac{\pi}{2} \in D_f$, *also* $f\left(\frac{\pi}{2}\right) = 2 + \left(\frac{\pi}{2} - \frac{\pi}{2}\right)^2 = 2$.

$$\lim_{h \to 0^-} f\left(\frac{\pi}{2} - h\right) = \lim_{h \to 0^-} \left(1 + \sin\left(\frac{\pi}{2} - h\right)\right) = 1 + \lim_{h \to 0^-} \cos h = 2,$$
$$\lim_{h \to 0^+} f\left(\frac{\pi}{2} + h\right) = \lim_{h \to 0^+} \left(2 + \left(\frac{\pi}{2} + h - \frac{\pi}{2}\right)^2\right) = 2 + \lim_{h \to 0^+} h^2 = 2.$$

Thus

$$\lim_{h \to 0^-} f\left(\frac{\pi}{2} - h\right) = \lim_{h \to 0^+} f\left(\frac{\pi}{2} + h\right) = f\left(\frac{\pi}{2}\right).$$

Hence the function $f(x)$ *is continuous at* $x = \frac{\pi}{2}$.

Again

$$\lim_{h \to 0^-} \frac{f\left(\frac{\pi}{2}-h\right)-f\left(\frac{\pi}{2}\right)}{-h} = \lim_{h \to 0^-} \frac{1+\cos h-2}{-h} = \lim_{h \to 0^-} \frac{1-\cos h}{h} = 0,$$
$$\lim_{h \to 0^+} \frac{f\left(\frac{\pi}{2}+h\right)-f\left(\frac{\pi}{2}\right)}{h} = \lim_{h \to 0^+} \frac{2+h^2-2}{h} = \lim_{h \to 0^+} \frac{h^2}{h} = 0.$$

$Lf'\left(\frac{\pi}{2}\right) = Rf'\left(\frac{\pi}{2}\right)$, *that is* $f'\left(\frac{\pi}{2}\right)$ *exists.*

Hence $f'\left(\frac{\pi}{2}\right)$ *is differentiable at* $x = \frac{\pi}{2}$.

Chapter 4 Exercise

1. Define following.
   (a). Continuous function
   (b). Discontinuous function
   (c). Cauchy definition of discontinuity
2. Show that the function

$$f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0, \\ 0 & x = 0 \end{cases}$$

   is discontinuous at $x = 0$, draw the function.
3. Show that $f(x) = \cos\frac{\pi}{x}$ has a discontinuity at $x = 0$.
4. A function defined as follows:

$$f(x) = \begin{cases} 5x - 4, & 0 < x \leq 1 \\ 4x^2 - 3x & 1 < x < 2 \end{cases}$$

   Discuss whether the function is continuous at $x = 1$.
5. A function defined as follows:

$$f(x) = \begin{cases} 1 + \sin x, & 0 \leq x \leq \frac{\pi}{2}, \\ 2 + \left(x - \frac{\pi}{2}\right)^2, & \frac{\pi}{2} \leq x \leq \infty \end{cases}$$

   Discuss the continuity and differentiability of the function at $x = \frac{\pi}{2}$.
6. A function defined as follows:

$$f(x) = \begin{cases} \frac{1}{2}\left(b^2 - a^2\right), & 0 < x \leq a, \\ \frac{1}{2}b^2 - \frac{x^2}{6} - \frac{a^3}{3x}, & a < x \leq b, \\ \frac{1}{3}\frac{\left(b^3 - a^3\right)}{x}, & x > b. \end{cases}$$

   Prove that $f(x)$ and $f'(x)$ are continuous but $f''(x)$ is discontinuous.

Chapter 5 Computation of Derivatives

Introduction

- *Derivatives of Polynomials*
- *The Product and Quotient Rules*
- *Composite Functions and the Chain Rule*
- *Derivatives of Exponential*
- *Derivatives of Logarithm*
- *Trigonometric Derivatives*
- *Derivatives of Inverse Trigonometric Function*
- *Logarithmic Differentiation*
- *Derivatives of Hyperbolic Function*
- *Derivatives of Inverse Hyperbolic Function*
- *Derivatives of Parametric Equation*
- *Implicit Functions*

5.1 Derivatives of Polynomials

As we know, the process of finding the derivative of a function is called differentiation. We already know the definition of derivatives

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}. \tag{5.1}$$

We have seen that this approach is rather slow and clumsy. Our purpose in the present chapter is to develop a small number of formal rules that will enable us to differentiate large classes of functions quickly, by purely mechanical procedures.

Theorem 5.1

If c is a constant then

$$\frac{dc}{dx} = 0. \tag{5.2}$$

Proof Let $y = f(x) = c$, where c is a constant.

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = 0.$$

Example 5.1

$$\frac{d5}{dx} = 0; \qquad \frac{d10!}{dx} = 0.$$

Theorem 5.2

$$\frac{d}{dx}x^n = nx^{n-1}. \tag{5.3}$$

Proof If n is positive and $y = f(x) = x^n$ then

$$\begin{aligned}
\frac{dy}{dx} = &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{(x+h)^n - x^n}{h} \\
&= x^n \lim_{h\to 0}\frac{(1+\frac{h}{x})^n - 1}{h} \\
&= x^n \lim_{h\to 0}\frac{\left(1 + n\frac{h}{x} + \frac{n(n-1)}{2}\left(\frac{h}{x}\right)^2 + \dots\right) - 1}{h} \\
&= x^n \lim_{h\to 0}\left(\frac{n}{x} + \frac{n(n-1)}{2}\left(\frac{h}{x^2}\right) + \dots\right) \\
\therefore \frac{d}{dx}x^n &= nx^{n-1}.
\end{aligned}$$

Though the theorem 5.2 is proved for integer, it is true for all rational number.

Example 5.2

$$\frac{dx^2}{dx} = 2x; \qquad \frac{dx^4}{dx} = 4x^3.$$

Theorem 5.3

If $u(x)$, and $v(x)$ are two derivable function of x then

$$\frac{d(u \pm v)}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}. \tag{5.4}$$

♡

Proof Let $y = f(x) = u(x) \pm v(x)$ then

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{(u(x+h) \pm v(x+h)) - (u(x) \pm v(x))}{h} \\
&= \lim_{h\to 0}\frac{u(x+h)-u(x)}{h} \pm \lim_{h\to 0}\frac{v(x+h)-v(x)}{h} \\
\therefore \frac{d(u\pm v)}{dx} &= \frac{du}{dx} \pm \frac{dv}{dx}.
\end{aligned}$$

Above theorem 5.3 can be extended for several variables.

Corollary 5.1

$$\frac{d}{dx}(u_1 \pm u_2 \pm \cdots \pm u_n) = \frac{du_1}{dx} \pm \frac{du_2}{dx} \pm \cdots \pm \frac{du_n}{dx}. \tag{5.5}$$

♡

Definition 5.1

A polynomial in x is a sum of constant multiples of powers of x in which each exponent is zero or a positive integer:

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$$

where n is an integer $(n > 0)$, a_0, a_1, a_2, ..., $a_n \in \mathbb{R}$, and $a_n \neq 0$ is known as the polynomial of degree n.

♣

Using eqs. (5.5)-(5.3) one can easily write

$$\frac{d}{dx}P(x) = a_n n x^{n-1} + a_{n-1}(n-1)x^{n-2} + a_{n-2}(n-2)x^{n-3} + \cdots + a_1.$$

Example 5.3

$$\begin{aligned}\frac{d}{dx}\left(15x^4 + 9x^3 - 7x^2 - 3x + 5\right) &= \frac{d}{dx}15x^4 + \frac{d}{dx}9x^3 - \frac{d}{dx}7x^2 - \frac{d}{dx}3x\frac{d}{dx}5 \\ &= 60x^3 + 27x^2 - 14x - 3.\end{aligned}$$

Theorem 5.4

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}} \tag{5.6}$$

♡

Proof

$$\frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{\frac{1}{2}} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}.$$

For any fractional factors eq. (5.3) can be used, and for radical function which can be converted to corresponding fractional factor.

5.2 The Product and Quotient Rules

Theorem 5.5

If $u(x)$, and $v(x)$ are two derivable function of x then

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}. \tag{5.7}$$

♡

Proof Let $y = f(x) = u(x)v(x)$ then

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0}\frac{u(x+h)v(x+h) - u(x)v(x)}{h} \\ &= \lim_{h\to 0}\frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h} \\ &= \lim_{h\to 0}\frac{u(x+h)\left(v(x+h) - v(x)\right) + v(x)\left(u(x+h) - u(x)\right)}{h} \\ &= \lim_{h\to 0}u(x+h)\lim_{h\to 0}\frac{v(x+h) - v(x)}{h} + \lim_{h\to 0}v(x)\lim_{h\to 0}\frac{u(x+h) - u(x)}{h} \\ \therefore \frac{d(uv)}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx}.\end{aligned}$$

Example 5.4

$$\begin{aligned}\frac{d}{dx}\left(\left(x^2 - 4x\right)\left(3x^2 + 2\right)\right) &= \left(x^2 - 4x\right)\frac{d}{dx}\left(3x^2 + 2\right) + \left(3x^2 + 2\right)\frac{d}{dx}\left(x^2 - 4x\right) \\ &= 6x\left(x^2 - 4x\right) + \left(3x^2 + 2\right)(2x - 4) \\ &= 6x^3 - 24x^2 + 6x^3 - 12x^2 + 4x - 8 \\ &= 12x^3 - 36x^2 + 4x - 8.\end{aligned}$$

Again

$$\frac{d}{dx}\left(\left(x^2-4x\right)\left(3x^2+2\right)\right) = \frac{d}{dx}\left(3x^4-12x^3+2x^2-8x\right) = 12x^3-36x^2+4x-8.$$

Corollary 5.2

If c is a constant, and $u(x)$ is a derivable function of x then

$$\frac{d(cu)}{dx} = c\frac{du}{dx}. \tag{5.8}$$

♡

Proof In eq. (5.7) replacing $v(x)$ by c,

$$\frac{d(cu)}{dx} = u\frac{dc}{dx} + c\frac{du}{dx} = c\frac{du}{dx}.$$

Example 5.5

$$\frac{d}{dx}5(3x^2+5x) = 5\frac{d}{dx}(3x^2+5x) = 5(6x+5) = 30x+25.$$

Theorem 5.6

If $u(x)$, and $v(x)$ are two derivable function of x and $v(x) \neq 0$ for all value of x then

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}. \tag{5.9}$$

♡

Proof Let $y = f(x) = \frac{u(x)}{v(x)}$ then

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{u(x+h)}{v(x+h)} - \frac{u(x)}{v(x)}\right) \\
&= \lim_{h\to 0}\frac{1}{h}\frac{u(x+h)v(x)-u(x)v(x+h)}{v(x+h)v(x)} \\
&= \lim_{h\to 0}\frac{1}{h}\frac{u(x+h)v(x)-u(x)v(x)+u(x)v(x)-u(x)v(x+h)}{v(x+h)v(x)} \\
&= \lim_{h\to 0}\frac{1}{h}\frac{v(x)\left(u(x+h)-u(x)\right)-u(x)\left(v(x+x)-v(x)\right)}{v(x+h)v(x)} \\
&= \frac{v(x)\left(\lim\limits_{h\to 0}\frac{u(x+h)-u(x)}{h}\right) - u(x)\left(\lim\limits_{h\to 0}\frac{v(x+x)-v(x)}{h}\right)}{\lim\limits_{h\to 0}v(x+h)v(x)} \\
\therefore \frac{d}{dx}\left(\frac{u}{v}\right) &= \frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}.
\end{aligned}$$

Example 5.6

$$\begin{aligned}
\frac{d}{dx}\frac{3x^2-1}{x^2+1} &= \frac{\left(x^2+1\right)\frac{d}{dx}\left(3x^2-1\right)-\left(3x^2-1\right)\frac{d}{dx}\left(x^2+1\right)}{\left(x^2+1\right)^2} \\
&= \frac{6x\left(x^2+1\right)-2x\left(3x^2-1\right)}{\left(x^2+1\right)^2} \\
&= \frac{6x^3+6x-6x^3+2x}{\left(x^2+1\right)^2} = \frac{8x}{\left(x^2+1\right)^2}
\end{aligned}$$

5.3 Composite Functions and the Chain Rule

We can find derivative of polynomial like

Example 5.7

$$\begin{aligned}\frac{d}{dx}(3x+2)^3 &= \frac{d}{dx}\left[(3x)^3+3(3x)^2 2+33x2^2+2^3\right]=\frac{d}{dx}\left(27x^3+54x^2+36x+8\right)\\ &= 81x^2+108x+36.\end{aligned}$$

but it will be cumbersome if same example become like $\frac{d}{dx}(3x+2)^{101}$. In this chapter, we will develop method for derivative of composite function.

Definition 5.2

Let $f(x)$ and $v(x)$ produce a new function $y(x)$, $y = f \circ v$ such that $y(x) = f(v(x))$ then y is called composition function *of f and v. Provided that $f \circ v \neq v \circ f$ i.e, $f(v(x)) \neq v(f(x))$.* ♣

The following theorem says how to find the derivative of composite function also known as the chain rule.

Theorem 5.7

Let $y = f(v)$, where $v = v(x)$, so is the function of x, then

$$\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{dx}, \tag{5.10}$$

where $f(v)$ and $v(x)$ are continuous. ♡

Theorem 5.8

Since $y = f(x) = f(v(x))$ then $f(x+h) = f(v(x+h))$. Let if $h \to 0$ then $k = v(x+h) - v(x) \to 0$. Now we have from the definition of derivative

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{f(v(x+h))-f(v(x))}{h}\\ &= \lim_{k\to 0}\frac{f(v(x+h))-f(v(x))}{v(x+h)-v(x)}\lim_{h\to 0}\frac{v(x+h)-v(x)}{h}\\ &= \lim_{k\to 0}\frac{f(v(x+h))-f(v(x))}{k}\lim_{h\to 0}\frac{v(x+h)-v(x)}{h}\\ \therefore \frac{dy}{dx} &= \frac{dy}{dv}\frac{dv}{dx}. \qquad \text{[Provided limits exist.]}\end{aligned}$$

♡

Example 5.8

$$\frac{d}{dx}(3x+2)^{101}$$

Let $u = 3x+2$ then $\frac{du}{dx} = 3$ we can write

$$\frac{du^{101}}{dx} = \frac{du^{101}}{du}\frac{du}{dx} = 101u^{100}3 = 303(3x+2)^{100}.$$

5.4 Derivatives of Exponential

Theorem 5.9

$$\frac{d}{dx}a^x = a^x \ln a. \tag{5.11}$$

Proof

$$\frac{d}{dx}a^x = \lim_{h\to 0}\frac{a^{x+h}-a^x}{h} = a^x \lim_{h\to 0}\frac{a^h-1}{h} = a^x \lim_{h\to 0}\frac{e^{h\ln a}-1}{h} = a^x \lim_{h\to 0}\frac{1+\frac{h\ln a}{1!}+\frac{(h\ln a)^2}{2!}+\cdots-1}{h}$$

$$\therefore \frac{d}{dx}a^x = a^x \ln a.$$

Corollary 5.3

$$\frac{d}{dx}e^x = e^x. \tag{5.12}$$

5.5 Derivatives of Logarithm

Theorem 5.10

$$\frac{d}{dx}\log_a x = \frac{1}{x}\log_a e. \tag{5.13}$$

Proof

$$\frac{d}{dx}\log_a x = \lim_{h\to 0}\frac{\log_a(x+h)-\log_a x}{h} = \lim_{h\to 0}\frac{1}{h}\log_a\left(1+\frac{h}{x}\right) = \frac{1}{x}\lim_{h\to 0}\frac{x}{h}\log_a\left(1+\frac{h}{x}\right)$$

Putting $z = \frac{x}{h}$, if $h \to 0$ then $z \to \infty$,

$$\frac{d}{dx}\log_a x = \frac{1}{x}\lim_{z\to\infty} z\log_a\left(1+\frac{1}{z}\right) = \frac{1}{x}\lim_{z\to\infty}\log_a\left(1+\frac{1}{z}\right)^z = \frac{1}{x}\log_a e. \quad [Eq.\ (3.3)]$$

$$\therefore \frac{d}{dx}\log_a x = \frac{1}{x}\log_a e.$$

Corollary 5.4

$$\frac{d}{dx}\ln x = \frac{1}{x}. \tag{5.14}$$

Problem 5.1 If $y = \ln\left[\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right]^{\frac{1}{2}}$ then show that $\frac{dy}{dx} = \frac{-1}{x\sqrt{1-x^2}}$.

Solution

$$y = \ln\left[\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right]^{\frac{1}{2}} = \frac{1}{2}\left[\ln\left(\sqrt{1+x}+\sqrt{1-x}\right) - \ln\left(\sqrt{1+x}-\sqrt{1-x}\right)\right]$$

Differentiating both side with respect to x

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{2}\left[\frac{-\frac{1}{2\sqrt{1+x}}+\frac{1}{2\sqrt{1-x}}}{\sqrt{1+x}+\sqrt{1-x}}+\frac{\frac{1}{2\sqrt{1+x}}+\frac{1}{2\sqrt{1-x}}}{\sqrt{1+x}-\sqrt{1-x}}\right] \\
&= \left[\frac{\frac{1}{2\sqrt{1+x}}\left(-\sqrt{1+x}+\sqrt{1-x}+\sqrt{1+x}+\sqrt{1-x}\right)+\frac{1}{2\sqrt{1-x}}\left(\sqrt{1+x}-\sqrt{1-x}+\sqrt{1+x}+\sqrt{1-x}\right.}{2(1+x-1+x)}\right. \\
&= \left[\frac{\frac{\sqrt{1-x}}{\sqrt{1+x}}+\frac{\sqrt{1+x}}{\sqrt{1-x}}}{4x}\right] \\
&= \frac{1}{4x}\frac{1-x+1+x}{\sqrt{1-x^2}} \\
&= \frac{1}{2x\sqrt{1-x^2}}
\end{aligned}$$

5.6 Trigonometric Derivatives

Theorem 5.11

$$\frac{d}{dx}\sin x = \cos x \tag{5.15}$$

$$\frac{d}{dx}\cos x = -\sin x \tag{5.16}$$

♡

Proof

$$\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{\sin(x)\cos h+\cos x\sin h-\sin x}{h} \\
&= -\sin x\lim_{h\to 0}\frac{1-\cos h}{h}+\cos x\lim_{h\to 0}\frac{\sin h}{h} = \cos x.
\end{aligned}$$

Similarly,

$$\frac{d}{dx}\cos x = -\sin x$$

Theorem 5.12

$$\frac{d}{dx}\tan x = \sec^2 x \tag{5.17}$$

$$\frac{d}{dx}\cot x = -\csc^2 x \tag{5.18}$$

♡

Proof

$$\begin{aligned}
\frac{d}{dx}\tan x &= \lim_{h\to 0}\frac{\tan(x+h)-\tan x}{h} = \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin(x)}{\cos(x)}\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos(x) - \cos(x+h)\sin x}{\cos(x+h)\cos(x)}\right] \\
&= \frac{1}{\cos^2(x)}\lim_{h\to 0}\left[\frac{\sin h}{h}\right] \\
\therefore \frac{d}{dx}\tan x &= \sec^2 x
\end{aligned}$$

Similarly,

$$\frac{d}{dx}\cot x = -\csc^2 x$$

Theorem 5.13

$$\frac{d}{dx}\sec x = \sec x \tan x \tag{5.19}$$

$$\frac{d}{dx}\csc x = -\csc x \cot x \tag{5.20}$$

♡

Proof

$$\begin{aligned}
\frac{d}{dx}\sec x &= \lim_{h\to 0}\frac{\sec(x+h)-\sec x}{h} = \lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\cos(x+h)} - \frac{1}{\cos(x)}\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{\cos(x) - \cos(x+h)}{\cos(x+h)\cos(x)}\right] \\
&= \frac{1}{\cos^2(x)}\lim_{h\to 0}\left[\frac{-2\sin(x+h/2)\sin(-h/2)}{h}\right] \\
&= \frac{1}{\cos(x)}\frac{\sin x}{\cos(x)}\lim_{h/2\to 0}\left[\frac{\sin(h/2)}{h/2}\right] \\
\therefore \frac{d}{dx}\tan x &= \sec x \tan x
\end{aligned}$$

Similarly,

$$\frac{d}{dx}\csc x = -\csc x \cot x$$

Example 5.9

$$\begin{aligned}
& y = \cos x^2 \\
\Longrightarrow\ & \frac{dy}{dx} = \frac{d}{dx}\cos x^2 = \frac{d}{dx^2}\cos x^2 \frac{dx^2}{dx} = -\sin x^2 \cdot 2x = -2x\sin x^2.
\end{aligned}$$

Example 5.10

$$\begin{aligned}
y &= x^n \sin ax \\
\Longrightarrow \frac{dy}{dx} &= \frac{d}{dx}x^n \sin ax = x^n \frac{d}{dx}\sin ax + \sin ax \frac{d}{dx}x^n \\
&= x^n \frac{d}{dax}\sin ax \frac{dax}{x} + nx^{n-1}\sin ax = ax^n \cos ax + nx^{n-1}\sin ax.
\end{aligned}$$

Example 5.11 Find dy/dx if $y = \sin(\cos x)$. Let $u = \cos x$ then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{d}{dx}\sin u \frac{d}{dx}\cos x = \cos u\,(-\sin x) = -\cos(\cos x)\sin x.$$

Example 5.12 Find dy/dx if $y = \sin[(1-x^2)/(1+x^2)]$. Let $u = (1-x^2)/(1+x^2)$ then

$$\begin{aligned} \frac{dy}{dx} &= \cos\left(\frac{1-x^2}{1+x^2}\right)\frac{d}{dx}\frac{1-x^2}{1+x^2} = \cos\left(\frac{1-x^2}{1+x^2}\right)\frac{-2x(1+x^2)-2x(1-x^2)}{(1+x^2)^2} \\ &= \frac{-4x}{(1+x^2)^2}\cos\left(\frac{1-x^2}{1+x^2}\right). \end{aligned}$$

5.7 Derivatives of Inverse Trigonometric Function

Theorem 5.14

$$\frac{d}{dx}\sin^{-1} = \frac{1}{\sqrt{1-x^2}} \tag{5.21}$$

$$\frac{d}{dx}\cos^{-1} = -\frac{1}{\sqrt{1-x^2}} \tag{5.22}$$

$$\frac{d}{dx}\tan^{-1} = \frac{1}{1+x^2} \tag{5.23}$$

$$\frac{d}{dx}\cot^{-1} = -\frac{1}{1+x^2} \tag{5.24}$$

$$\frac{d}{dx}\sec^{-1} = \frac{1}{x\sqrt{x^2-1}} \tag{5.25}$$

$$\frac{d}{dx}\csc^{-1} = -\frac{1}{x\sqrt{x^2-1}} \tag{5.26}$$

♡

5.8 Logarithmic Differentiation

Theorem 5.15

$$\frac{dy}{dx} = \frac{d}{dx}u(x)^{v(x)} = u^v\left(\frac{v}{u}\frac{du}{dx} + \frac{dv}{dx}\ln u\right) \tag{5.27}$$

♡

Proof Let $y = u(x)^{v(x)}$ taking $\ln$ on both side

$$\ln y = \ln u(x)^{v(x)} = v(x)\ln u(x)$$

Differentiating both side with respect to x

$$\begin{aligned} \frac{1}{y}\frac{dy}{dx} &= \frac{v}{u}\frac{du}{dx} + \frac{dv}{dx}\ln u \\ \therefore \frac{dy}{dx} &= u^v\left(\frac{v}{u}\frac{du}{dx} + \frac{dv}{dx}\ln u\right) \end{aligned}$$

Problem 5.2 Differentiate $y = x + x^x$ with respect to x.

Solution

$$y = x + x^x = x + e^{\ln x^x} = x + e^{x \ln x}$$

Taking derivative both side with respect to x

$$\frac{dy}{dx} = 1 + e^{x \ln x}(1 + \ln x) = 1 + x^x(1 + \ln x).$$

Problem 5.3 Differentiate $y = (\cot x)^{\sin x} + (\tan x)^{\cos x}$ with respect to x.

Solution

$$y = (\cot x)^{\sin x} + (\tan x)^{\cos x} = e^{\sin x \ln \cot x} + e^{\cos x \ln \tan x} \tag{5.28}$$

Differentiate both side with respect to x*,*

$$\begin{aligned}
\frac{dy}{dx} &= e^{\sin x \ln \cot x}\left(\frac{\sin x}{\cot x}(-\cot x \csc x) + \cos x \ln \cot\right) \\
&\quad + e^{\cos x \ln \tan x}\left(\frac{\cos x}{\tan x}\sec x \tan x - \cos x \ln \tan x\right) \\
\frac{dy}{dx} &= (\cot x)^{\sin x}(\cos x \ln \cot x - 1) + (\tan x)^{\cos x}(1 - \cos x \ln \tan x)
\end{aligned}$$

5.9 Derivatives of Hyperbolic Function

Theorem 5.16

$$\frac{d}{dx}\sinh x = \cosh x \tag{5.29}$$

$$\frac{d}{dx}\cosh x = \sinh x \tag{5.30}$$

$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x \tag{5.31}$$

$$\frac{d}{dx}\coth x = -\operatorname{csch}^2 x \tag{5.32}$$

$$\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x \tag{5.33}$$

$$\frac{d}{dx}\operatorname{csch} x = -\operatorname{csch} x \coth x \tag{5.34}$$

♡

5.10 Derivatives of Inverse Hyperbolic Function

Theorem 5.17

$$\frac{d}{dx}\sinh^{-1}x = \frac{1}{\sqrt{1+x^2}} \tag{5.35}$$

$$\frac{d}{dx}\cosh^{-1}x = \frac{1}{\sqrt{x^2-1}} \tag{5.36}$$

$$\frac{d}{dx}\tanh^{-1}x = \frac{1}{1-x^2} \tag{5.37}$$

$$\frac{d}{dx}\coth^{-1}x = -\frac{1}{x^2-1} \tag{5.38}$$

$$\frac{d}{dx}\operatorname{sech}^{-1}x = \frac{1}{x\sqrt{1-x^2}} \tag{5.39}$$

$$\frac{d}{dx}\operatorname{csch}^{-1}x = -\frac{1}{x\sqrt{x^2+1}} \tag{5.40}$$

♡

5.11 Derivatives of Parametric Equation

Definition 5.3

Let x and y both are function of t called parameter *i.e.* $x = x(t)$, *and* $y = y(t)$, *and the equations of* x, y *are called* parametric equations.

♣

Theorem 5.18

If x, and y are both parametric functions of t then

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt}. \tag{5.41}$$

♡

Example 5.13 Find $\frac{dy}{dx}$, where $x = \cos\theta$ and $y = \sin\theta$.

$$\frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta} = \frac{\cos\theta}{-\sin\theta} = -\cot\theta.$$

Problem 5.4 Find the slope of a tangent to the curve

$$x = a\,(t - \sin t)$$

$$y = a\,(1 - \cos t)\,.$$

at any point $0 \le t \le 2\pi$.

Solution

$$\frac{dx}{dt} = a\,(1 - \cos t)$$

$$\frac{dy}{dt} = a\,(1 + \sin t)$$

Hence,

$$\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{1+\sin t}{1-\cos t}$$

5.12 Implicit Functions

Definition 5.4

If an equation involving x, and y, the variable y is not given in terms of x, or it is not suitable to express in terms of x by solving the equation, then y is said to be implicit function. ♣

Example 5.14 Differentiate

$$ax^2 + 2hxy + by^2 + d = 0$$

with respect to x.

Differentiating every terms with respect to x,

$$\begin{aligned} & 2ax + 2hy + 2hx\frac{dy}{dx} + 2by\frac{dy}{dx} = 0 \\ \implies & (hx + by)\frac{dy}{dx} = -(ax + hy) \\ \implies & \frac{dy}{dx} = \frac{-ax - hy}{hx + by}. \end{aligned}$$

Problem 5.5 Find $\frac{dy}{dx}$, where $y = \tan^{-1} y \ln \sec^2 x^2$.

Solution

$$y = \tan^{-1} y \ln \sec^2 x^2$$

Differentiating both side with respect to x,

$$\begin{aligned} \implies & \frac{dy}{dx} = \frac{1}{1+y^2}\frac{dy}{dx}\ln \sec^2 x^2 + \tan^{-1} y \frac{1}{\sec^2 x^2} 2\sec x^2 2x \\ \implies & \frac{dy}{dx}\left(1 - \frac{\ln \sec^2 x^2}{1+y^2}\right) = 4x \tan^{-1} y \cos x^2 \\ \implies & \frac{dy}{dx} = 4x \tan^{-1} y \cos x^2 \left(1 - \frac{\ln \sec^2 x^2}{1+y^2}\right)^{-1} \end{aligned}$$

Problem 5.6 Find $\frac{dy}{dx}$, where $\tan y = e^{\cos 2x} \sin x$.

Solution

$$\tan y = e^{\cos 2x} \sin x$$

Differentiating every terms with respect to x,

$$\begin{aligned} \sec y \tan y \frac{dy}{dx} &= e^{\cos 2x}(-2\sin 2x)\sin x + e^{\cos 2x}\cos x \\ \frac{dy}{dx} &= \frac{1}{\sec y \tan y} e^{\cos 2x}(-2\sin 2x \sin x + \cos x) \\ \frac{dy}{dx} &= \frac{\cos y}{e^{\cos 2x}\sin x} e^{\cos 2x}\left(-4\sin^2 x \cos x + \cos x\right) \\ \frac{dy}{dx} &= \cos y \cot x \left(1 - 4\sin^2 x\right) \end{aligned}$$

Chapter 5 Exercise

1. Define derivative of a function.

2. Define polynomial.
3. What do you mean by differentiation.
4. Write product rule for differentiation.
5. Write quotient rule for differentiation.
6. Find the derivative of following functions by using the definition of derivative
   (a). x^n,
   (b). $\sin x$,
   (c). $\cos x$,
   (d). $\tan x$,
   (e). $\cot x$,
   (f). $\sec x$,
   (g). $\csc x$,
   (h). a^x,
   (i). $\ln x$.
7. Differentiate each function two ways, and verify that your answers agree.
   (a). $(2x-6)(3x^2+9)$
   (b). $(x-1)\left(x^4+x^3+x^2+x+1\right)$
   (c). $(x^3-3x)(x^2+5)$
   (d). $(x^4+1)(x^4-1)$
8. Differentiate each function and simplify your answer as much as possible.
   (a). $\frac{x-1}{x+1}$
   (b). $\frac{4x-x^4}{x^3+2}$
   (c). $\frac{\frac{1}{x}-\frac{3}{x^2}}{\frac{5}{x^3}-\frac{7}{x^4}}$
   (d). $\frac{1}{1-2x^{-2}}$
9. Find $\frac{dy}{dx}$, where
   (a). $y=x+x^x$,
   (b). $\sin y = x\sin(x+y)$,
   (c). $\tan y = e^{\cos 2x}\sin x$,
   (d). $\tan y = \frac{2t}{1-t^2}$; $\sin x = \frac{2t}{1+t^2}$.
   (e). $y=\tan^{-1} y \ln \sec^2 x^2$,
   (f). $y=(\cot x)^{\sin x}+(\tan x)^{\cos x}$.
10. If $y=\frac{\tan x}{\ln x + x^{\frac{1}{2}}}$ then find $\frac{dy}{dx}$.
11. If $y=\ln\left[\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right]^{\frac{1}{2}}$ then show that $\frac{dy}{dx}=\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$.
12. Differentiate $\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}$ with respect to $\sqrt{1-x^4}$.
13. Differentiate $\ln\tan\sqrt{x-1}$ with respect to $\sqrt{x-1}$.
14. Find the slope of a tangent to the curve

$$x=a(t-\sin t)$$
$$y=a(1-\cos t).$$

at any point $0 \le t \le 2\pi$.

Chapter 6 Successive Differentiation

Introduction

- *TO DO*

Problem 6.1 If $y = x^3 \sin 2x$ find y_n.

Solution *TO DO*

$$\begin{aligned} & y = x^3 \sin 2x \\ \Longrightarrow \quad & y_n = \left(x^3 \sin 2x\right)_n \\ \Longrightarrow \quad & y_n = \end{aligned} \tag{6.1}$$

Chapter 6 Exercise

1. Find y_n for following functions
   (a). $y = x^3 \sin 2x$,
   (b). $y = e^{ax} \{a^2x^2 - 2nax + n(n+1)\}$,
   (c). $y = (x^3 + 2x^2 + x + 1)\, a^x$,
2. If

$$y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$$

then proved that

$$\left(x^2 - 1\right) y_{n+2} + (2x+1)\, x y_{n+1} \left(n^2 - m^2\right) y_n.$$

Appendix A Trigonometric Identities

Table A.1: Trigonometric Identities-Sum of Angles

$\sin(A \pm B)$	$\sin A \cos B \pm \cos A \sin B$
$\cos(A \pm B)$	$\cos A \cos B \mp \sin A \sin B$
$\tan(A \pm B)$	$\frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

Table A.2: Trigonometric Identities-Product Rules

$2\sin A \cos B$	$\sin(A+B) + \sin(A-B)$
$2\cos A \sin B$	$\sin(A+B) - \sin(A-B)$
$2\cos A \cos B$	$\cos(A+B) + \cos(A-B)$
$2\sin A \sin B$	$\cos(A-B) - \cos(A+B)$

Table A.3: Trigonometric Identities-Sum Rules

$\sin C + \sin D$	$2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$
$\sin C - \sin D$	$2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$
$\cos C + \cos D$	$2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$
$\cos C - \cos D$	$-2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

Table A.4: Trigonometric Identities-Double Angles

$\sin 2A$	$2\sin A \cos A$
$\cos 2A$	$\cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$
$\tan 2A$	$\frac{2\tan A}{1-\tan^2 A}$

Table A.5: Trigonometric Identities-Triple Angles

$\sin 3A$	$3\sin A - 4\sin^3 A$
$\cos 3A$	$4\cos^3 A - 3\cos A$

Table A.6: Trigonometric Identities-Fraction Angles

$\sin A$	$2\sin\frac{A}{2}\cos\frac{A}{2}$
$\cos A$	$\cos^2\frac{A}{2} - \sin^2\frac{A}{2} = 2\cos^2\frac{A}{2} - 1 = 1 - 2\sin^2\frac{A}{2}$
$\sin A$	$3\sin\frac{A}{3} - 4\sin^3\frac{A}{3}$
$\cos A$	$4\cos^3\frac{A}{3} - 3\cos\frac{A}{3}$

Appendix B Table of Fundamental Differential & Integral Formulae

Appendix C Standard Integral Formulae

Appendix D List of Corrector of this Note

1. Rajika Chanda Dia, Eq. (2.3), Aeronautical & Aviation Sc. & Eng., 2022, CATECH.
2. Rajika Chanda Dia, Ex. 3.12, Aeronautical & Aviation Sc. & Eng., 2022, CATECH.

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